Frequency Response
주파수 응답

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전달함수 (Transfer Function)

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t) \qquad H(s) = \frac{Y(s)}{X(s)} = \left.\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}\right|_{\text{all ICs} = 0}$$

$$i_C(t) \longrightarrow \boxed{H_C(s)} \longrightarrow v_C(t)$$

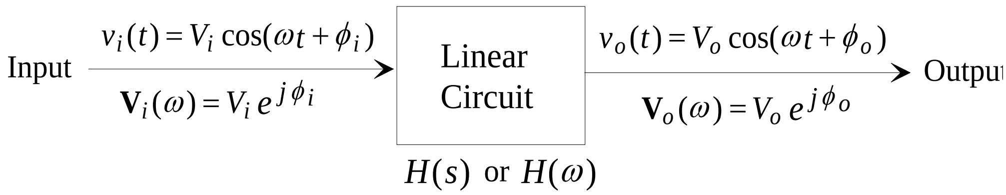
$$H_C(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{Cs} (= Z_C(s))$$

$$i_L(t) \longrightarrow \boxed{H_L(s)} \longrightarrow v_L(t)$$

$$H_L(s) = \frac{V_L(s)}{I_L(s)} = Ls(= Z_L(s))$$

주파수 응답 함수 (Frequency Response Function)

- 주파수 응답 (frequency response): 정현파 입력에 대한 시스템의 정상 상태 응답
- 주파수 응답 함수(Frequency Response Function; FRF)



- 전달함수: $H(s) = \dfrac{V_o(s)}{V_i(s)} \Rightarrow V_o(s) = H(s)V_i(s)$
- 주파수 응답함수: $H(\omega) = \dfrac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} \Rightarrow \mathbf{V}_o(\omega) = H(\omega)\mathbf{V}_i(\omega)$

FRF의 크기와 위상

$$H(\omega) = |H(\omega)|e^{j\phi_H} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{V_o e^{j\phi_o}}{V_i e^{j\phi_i}}$$

- 크기(magnitude) 또는 이득(gain) :

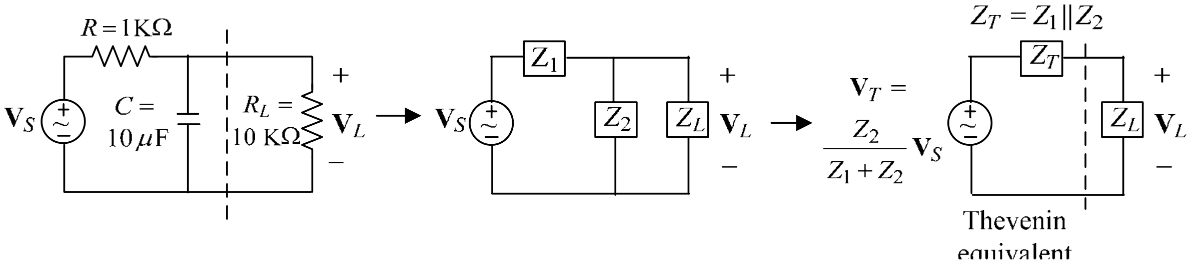
$$|H(\omega)| = V_o / V_i$$

- 위상(phase):

$$\phi_H = \phi_o - \phi_i$$

Example

- 주파수 응답 함수 $H(\omega)=\mathbf{V}_L(\omega)/\mathbf{V}_S(\omega)$ 를 계산하라



$$\mathbf{V}_L=\frac{Z_L}{Z_T+Z_L}\mathbf{V}_T=\frac{Z_L}{(Z_1\|Z_2)+Z_L}\frac{Z_2}{Z_1+Z_2}\mathbf{V}_S=\frac{Z_L(Z_1+Z_2)}{Z_1Z_2+Z_L(Z_1+Z_2)}\frac{Z_2}{Z_1+Z_2}\mathbf{V}_S=\frac{Z_LZ_2}{Z_L(Z_1+Z_2)+Z_1Z_2}\mathbf{V}_S$$

$$H(\omega)=\frac{\mathbf{V}_L}{\mathbf{V}_S}=\frac{Z_LZ_2}{Z_L(Z_1+Z_2)+Z_1Z_2}=\frac{10000\times\frac{1}{j\omega\times10^{-5}}}{10000\times\left(1000+\frac{1}{j\omega\times10^{-5}}\right)+1000\times\frac{1}{j\omega\times10^{-5}}}=\frac{100}{j\omega+110}$$

$$|H(\omega)|=\frac{100}{\sqrt{\omega^2+110^2}}\quad \& \quad \angle H(\omega)=\phi_H=0-\tan^{-1}\left(\frac{\omega}{110}\right)=-\tan^{-1}\left(\frac{\omega}{110}\right)$$

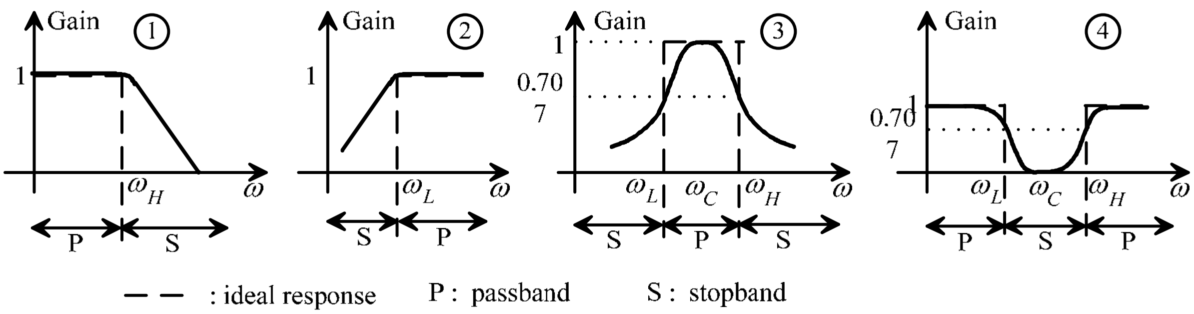
필터 (Filter)

- 전기 필터: 지정된 대역의 주파수는 통과시키고 이 대역 외의 주파수는 감쇠시키는 주파수 선택적인 회로

- 필터의 종류
  - 아날로그 필터 또는 디지털 필터
  - 수동(passive) 필터 또는 능동(active) 필터

- 수동 필터
  - 저항, 커패시터 및 인덕터로 구성된 필터.
  - 출력 신호의 크기 < 입력 신호의 크기

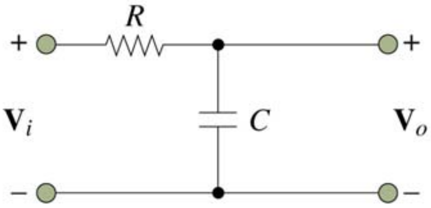
주파수 특성에 따른 필터의 종류



1. 저역 통과 필터(low-pass filter), 2. 고역 통과 필터(high-pass filter),
3. 대역 통과 필터(band-pass filter), 4. 대역 제거 필터(band-reject filter)

저역통과 필터 (Low-Pass Filter; LPF)

RC low-pass filter. The circuit preserves lower frequencies while attenuating the frequencies above the cutoff frequency $\omega_0 = 1/RC$. The voltages $\mathbf{V}_i$ and $\mathbf{V}_o$ are the filter input and output voltages, respectively.



$$\mathbf{V}_o(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \mathbf{V}_i(\omega) = \frac{1}{1 + j\omega RC} \mathbf{V}_i(\omega) \Rightarrow H(\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + (\omega/\omega_o)^2}}$$

$$\angle H(\omega) = \phi(\omega) = -\tan^{-1}(\omega RC) = -\tan^{-1}(\omega/\omega_o)$$

$\omega_o = 1/RC$: 차단 주파수(cutoff frequency, break frequency or corner frequency)

Bode Plot

- 크기(Magnitude):

$$\omega \to 0 \;\Rightarrow\; |H(\omega)| \to 1 \text{ (i.e., } V_o \to V_i \;\because 1/j\omega C \to \infty \text{ as } \omega \to 0 \;\Rightarrow\; \text{capacitor: open-circuit)}$$

$$\omega = \omega_o \;\Rightarrow\; |H(\omega)| = 1/\sqrt{2}$$

$$\omega \to \infty \;\Rightarrow\; |H(\omega)| \to 0 \text{ (i.e., } V_o \to 0 \;\because 1/j\omega C \approx 0 \text{ as } \omega \to \infty \;\Rightarrow\; \text{capacitor: short-circuit)}$$

- 위상 (Phase): $\omega \to 0 \Rightarrow \phi(\omega) \to 0^\circ, \omega = \omega_o \Rightarrow \phi(\omega) = -45^\circ, \omega \to \infty \Rightarrow \phi(\omega) \to -90^\circ$
- 데시벨(Decibel): $\left|\frac{\mathbf{V}_o}{\mathbf{V}_i}\right|_{\mathrm{dB}} = 20\log_{10}\left|\frac{\mathbf{V}_o}{\mathbf{V}_i}\right|$ (dB)

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_o)^2}} \;\Rightarrow\; 20\log_{10}|H(\omega)| = 20\log_{10}\frac{1}{\sqrt{1+(\omega/\omega_o)^2}} = -10\log_{10}\left|1+(\omega/\omega_o)^2\right|$$

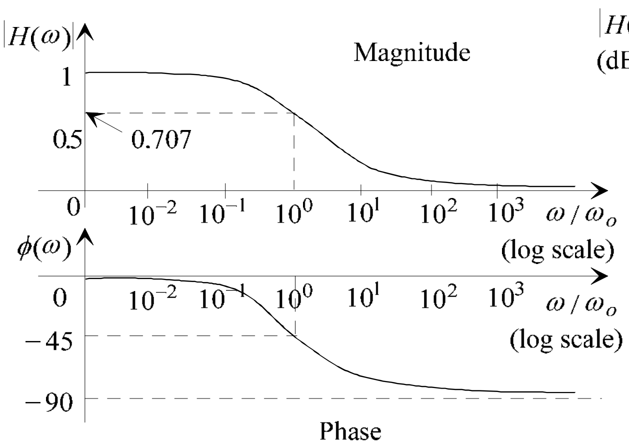
$$\omega \to 0 \;\Rightarrow\; 20\log_{10}|H(\omega)| = -10\log_{10}1 = 0$$

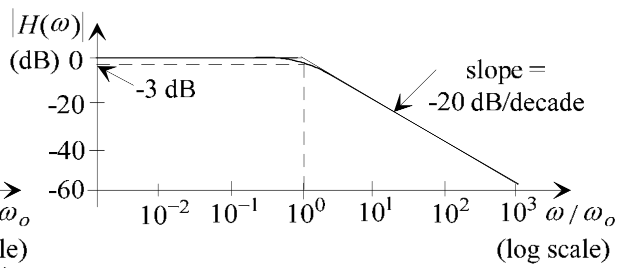
$$\omega = \omega_o \;\Rightarrow\; 20\log_{10}|H(\omega)| = -10\log_{10}2 \approx -3$$

$$\omega \to \infty \;\Rightarrow\; 20\log_{10}|H(\omega)| \approx -10\log_{10}\left|(\omega/\omega_o)^2\right| = -20\log_{10}|\omega/\omega_o|$$

저역통과 필터의 특성

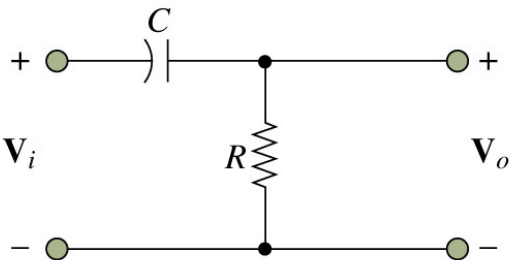
- 저주파 대역에서는 신호를 통과시킨다. $(\omega << \omega_o = 1/RC)$
- 고주파 대역에서는 신호를 제거하거나 감쇠시킨다. $(\omega << \omega_o = 1/RC)$
- LPF의 특성은 차단 주파수의 값(R과 C에 의존함)을 변화시킴으로써 변하게 된다





고역통과 필터 (High-Pass Filter; HPF)

RC high-pass filter. The circuit preserves higher frequencies while attenuating the frequencies below the cutoff frequency $\omega_0 = 1/RC$.



$$\mathbf{V}_o(\omega) = \frac{R}{R + \frac{1}{j\omega C}}\mathbf{V}_i(\omega) = \frac{j\omega RC}{1 + j\omega RC}\mathbf{V}_i(\omega) \Rightarrow \mathbf{H}(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} = \frac{\omega/\omega_o}{\sqrt{1 + (\omega/\omega_o)^2}}$$

$$\angle H(\omega) = \phi(\omega) = \tan^{-1}(\omega RC/0) - \tan^{-1}(\omega RC) = 90^\circ - \tan^{-1}(\omega/\omega_o)$$

$$\omega_o = 1/RC$$

Bode Plot

- 크기

$$\omega \to 0 \Rightarrow H(\omega) \to 0, \omega = \omega_o \Rightarrow H(\omega) = 1/\sqrt{2}, \omega \to \infty \Rightarrow H(\omega) \to 1$$

$$\omega \to 0 \Rightarrow H(\omega) \to 0 \text{ (i.e., } V_o \to 0 \ \because 1/j\omega C \to \infty \text{ as } \omega \to 0 \Rightarrow \text{capacitor : open - circuit)}$$

$$\omega = \omega_o \Rightarrow H(\omega) = 1/\sqrt{2}$$

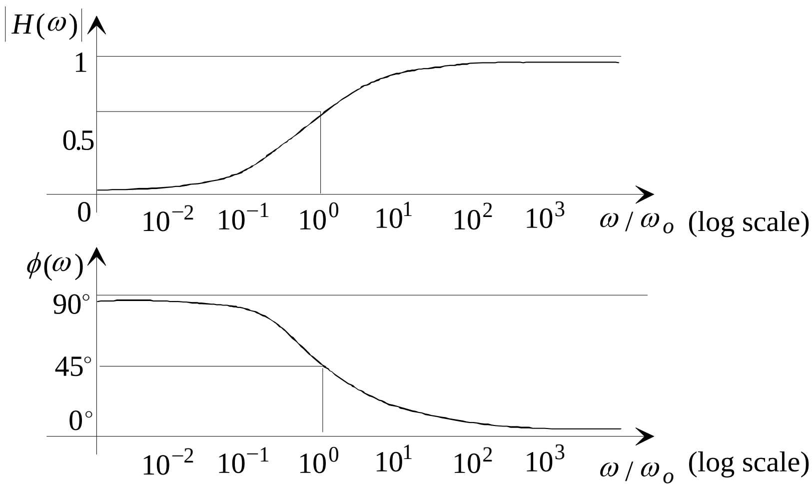
$$\omega \to \infty \Rightarrow H(\omega) \to 0 \text{(i.e., } V_o \to V_i \ \because 1/j\omega C \to 0 \text{ as } \omega \to \infty \Rightarrow \text{capacitor : short - circuit)}$$

- 위상

$$\omega \to 0 \Rightarrow \phi(\omega) \to 90^\circ, \omega = \omega_o \Rightarrow \phi(\omega) = 45^\circ, \omega \to \infty \Rightarrow \phi(\omega) \to 0^\circ$$

고역통과 필터의 특성

- 저주파 대역에서는 신호를 제거하거나 감쇠시킨다. $(\omega << \omega_o = 1/RC)$
- 고주파 대역에서는 신호를 통과시킨다. $(\omega >> \omega_o = 1/RC)$
- HPF의 특성은 차단 주파수의 값(R과 C에 의존함)을 변화시킴으로써 변하게 된다.

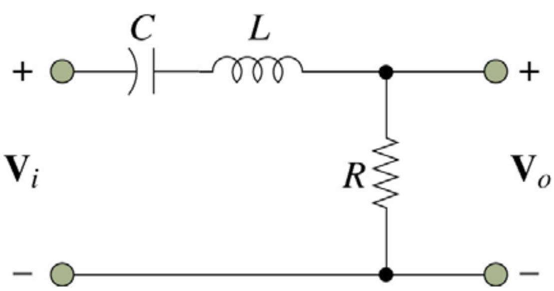


대역통과 필터 (Band-Pass Filter; BPF)

$$\mathbf{V}_o(\omega) = \frac{R}{R + 1/j\omega C + j\omega L}\mathbf{V}_i(\omega) = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC}\mathbf{V}_i(\omega)$$

$$H(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC} \overset{Let}{=} \frac{jRC\omega}{(1 + j\omega/\omega_L)(1 + j\omega/\omega_H)}$$

RLC bandpass filter. The circuit preserves frequencies within a band.



$$\frac{1}{\omega_L} + \frac{1}{\omega_H} = RC \; \& \; \frac{1}{\omega_L \omega_H} = LC$$

ω_L : low cutoff frequency
ω_H : high cutoff frequency

$\Delta\omega = \omega_H - \omega_L$: 대역폭(bandwidth)

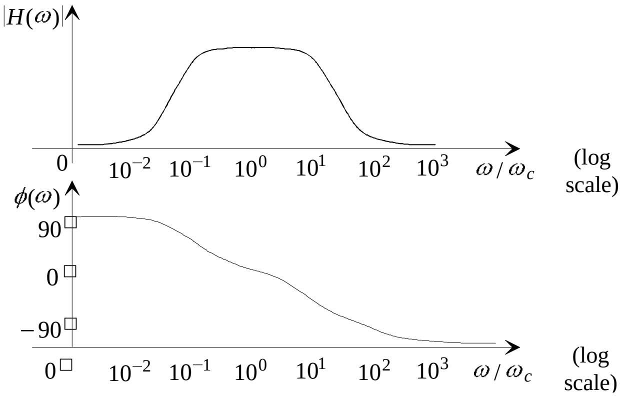
$\omega_c = \sqrt{\omega_L \omega_H}$: 중심 주파수(center frequency)

$$|H(\omega)| = \frac{RC\omega}{\sqrt{1 + (\omega/\omega_L)^2}\sqrt{1 + (\omega/\omega_H)^2}}$$

$$\angle H(\omega) = \phi(\omega) = 90^\circ - \tan^{-1}(\omega/\omega_H) - \tan^{-1}(\omega/\omega_L)$$

Bode Plot

- 크기: $\omega \to \infty \Rightarrow |H(\omega)| \to 0 (\text{i.e.}, V_o \to 0 \;\because\; j\omega L \to \infty \text{ as } \omega \to \infty \;\Rightarrow \text{inductor : open - circuit})$
- 위상: $\omega \to 0 \Rightarrow \phi(\omega) \to 90^\circ, \omega = \omega_c \Rightarrow \phi(\omega) = 0^\circ, \omega \to \infty \Rightarrow \phi(\omega) \to -90^\circ$
- 대역 통과 필터의 특성
  - 통과 대역에서는 신호를 통과시킨다 $(\omega_L < \omega < \omega_H)$
  - 대역 통과 필터의 특성은 *R*, *L*과 *C*의 값에 의해서 변화시킬 수 있다.



공진과 대역폭

$$\frac{V_o}{V_i}(j\omega) = \frac{j\omega CR}{LC(j\omega)^2 + j\omega CR + 1}$$
$$= \frac{(2\zeta/\omega_n)j\omega}{(j\omega/\omega_n)^2 + (2\zeta/\omega_n)j\omega + 1}$$
$$= \frac{(1/Q\omega_n)j\omega}{(j\omega/\omega_n)^2 + (1/Q\omega_n)j\omega + 1}$$

$$\omega_n = \sqrt{\frac{1}{LC}} = \text{natural or resonant frequency}$$

$$Q = \frac{1}{2\zeta} = \frac{1}{\omega_n CR} = \frac{1}{R}\sqrt{\frac{L}{C}} = \text{quality factor}$$

$$\zeta = \frac{1}{2Q} = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{damping ratio}$$

$$B = \frac{\omega_n}{Q} = \text{대역폭}$$