ME2025 Digital Control

Discrete Equivalents to Continuous Transfer Functions

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Approaches to Digital Control

1. Emulation
   - Emulation of continuous time designs
   - Do all design for continuous time systems, then replace the resulting controller with its discrete equivalent
2. Direct Methods
   - Add samplers, holds (A/D and D/A) to make subsystems digital
   - Then do design using discrete time systems

Discrete Equivalents to Continuous Transfer Functions

- Many ways to do this:
  - Impulse Invariance
  - Pole-Zero Matching
  - Numerical Integration Methods
  - Hold Equivalents

Impulse Invariant Laplace to z-Transform Conversion

This method is easy. Impulse invariance means that digital system impulse response exactly equals samples of the analog impulse response: $h[n]=h_a(nT)$ for all n

Causal LTI continuous time system

$$H_c(s)=\frac{b_o+b_1 s+b_2 s^2+\ldots+b_p s^p}{1+a_1 s+a_2 s^2+\ldots+a_p s^p}=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\ldots+\frac{A_p}{s-s_p}$$

This implies $h_c(t)=(A_1 e^{s_1 t}+A_2 e^{s_2 t}+\ldots A_p e^{s_p t})u(t)$

Impulse Invariant Laplace to z-Transform Conversion

$$h_c(t) = (A_1 e^{s_1 t} + A_2 e^{s_2 t} + \ldots A_p e^{s_p t})u(t)$$

For impulse invariance we want to match at sampling instants:

$$h[n] = h_c(nT) = (A_1 e^{s_1 nT} + A_2 e^{s_2 nT} + \ldots + A_p e^{s_p nT})u[n]$$

Since $A_k e^{s_k T n} 1[n] \leftrightarrow \dfrac{A_k z}{z - e^{s_k T}}$ where $|z| > |e^{s_k T}|$

$$\boxed{H(z) = \sum_{k=1}^{p} \frac{A_k z}{z - e^{s_k T}}}$$ where $|z| > \max_k |e^{s_k T}|$

Impulse Invariant Laplace to z-Transform Conversion

$$\boxed{H(z) = \sum_{k=1}^{p} \frac{A_k z}{z - e^{s_k T}}}$$ where $|z| > \max_k |e^{s_k T}|$

Stable s-plane poles will map to stable z-plane poles here

Zero mapping is much more complicated

Aliasing is a problem with this approach

Pole-Zero Matching

- Rule 1: All poles of H(s) mapped via $z = e^{sT}$ into poles of $H_{zp}(z)$
- Rule 2: All *finite* zeros of H(s) mapped via $z = e^{sT}$ into zeros of $H_{zp}(z)$ [that is, zeros that explicitly appear in H(s)]

Pole-Zero Matching

- Rule 3a: Map the *infinite* zeros of H(s) to $z = -1$ [that is, if H(s) has n poles and m zeros, then $H_{zp}(z)$ has $n-m$ zeros at $z = -1$] Highest frequency
- Rule 3b: if a one step delay in the system response is desired, then only $(n-m-1)$ zeros at $z = -1$

Pole-Zero Matching

- Rule 4: Match gains of $H(s)$ and $H_{zp}(z)$ at a critical location -- usually at $s = 0$, which would make

$$H(s)|_{s=0} = H_{zp}(z)|_{z=1}$$

Pole-Zero Matching

- For rules 1 and 2: map $\mathbf{s = -a}$ into $\mathbf{z = e^{-aT}}$ and map $\mathbf{s = -a+jb}$ into $\mathbf{z = re^{j\theta T}}$ where $\mathbf{r = e^{-aT}}$ and $\theta = \mathbf{bT}$

Pole-Zero Matching

Example

$$H(s) = \frac{a}{s+a}$$

Pole at s = -a maps into pole at e^{-aT}

The zero at s = ∞ will map into a zero at z = -1.

The gain of H(s) at s=0 is 1. To match this gain in H(z) at z=1 requires a gain of $\frac{1-e^{-aT}}{2}$

Result is

$$H_{zp}(z) = \frac{(z+1)(1-e^{-aT})}{2(z-e^{-aT})}$$

Or using rule 3(b)

$$H_{zp}(z) = \frac{(1-e^{-aT})}{(z-e^{-aT})}$$

Discrete Equivalents via Numerical Integration

$$\dot{u} + au = ae$$

- Solving this differential equation numerically is like approximating the following integral:

$$u(t) = \int_0^t [-au(\tau) + ae(\tau)]d\tau,$$

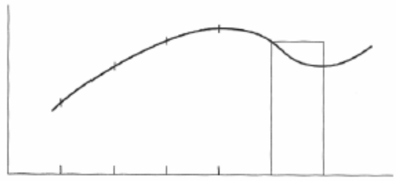
$$u(kT) = \int_0^{kT-T} [-au + ae]d\tau + \int_{kT-T}^{kT} [-au + ae]d\tau$$

$$= u(kT-T) + \left\{ \begin{array}{l} \text{area of } -au + ae \\ \text{over } kT - T \le \tau < kT \end{array} \right.$$

- Many different ways to approximate the last term

Discrete Equivalents via Numerical Integration

- Forward Rectangular Rule (Euler's Rule)

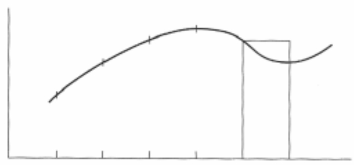


Forward Rectangular Rule

$$
\begin{aligned}
u_1(kT) &= u_1(kT-T) + T[-au_1(kT-T) + ae(kT-T)] \\
&= (1-aT)u_1(kT-T) + aTe(kT-T)
\end{aligned}
$$

Discrete Equivalents via Numerical Integration

- Forward Rectangular Rule (Euler's Rule)



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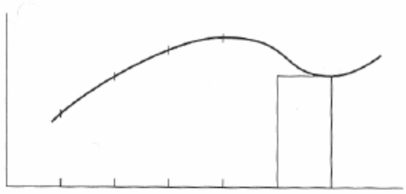
\end{aligned}
$$

•Transfer Function using this:

$$
\begin{aligned}
H_F(z) &= \frac{aTz^{-1}}{1-(1-aT)z^{-1}} \\
&= \frac{a}{(z-1)/T + a}
\end{aligned}
$$

Discrete Equivalents via Numerical Integration

- Backward Rectangular Rule



Backward Rectangular Rule

$$
\begin{aligned}
u_2(kT) &= u_2(kT-T) + T[-au_2(kT) + ae(kT)] \\
&= \frac{u_2(kT-T)}{1+aT} + \frac{aT}{1+aT}e(kT)
\end{aligned}
$$

Discrete Equivalents via Numerical Integration

- Backward Rectangular Rule

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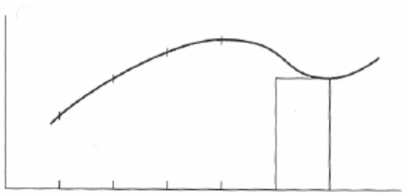


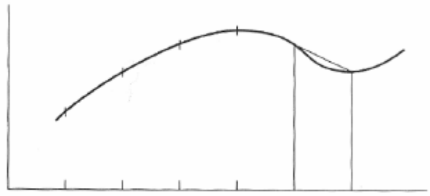
Backward Rectangular Rule

- Transfer Function using this:

$$
\begin{aligned}
H_B(z) &= \frac{aT}{1+aT}\frac{1}{1-z^{-1}/(1+aT)} = \frac{aTz}{z(1+aT)-1} \\
&= \frac{a}{(z-1)/Tz + a} \qquad \text{(backward rectangular rule)}
\end{aligned}
$$

Discrete Equivalents via Numerical Integration

- Trapezoid Rule

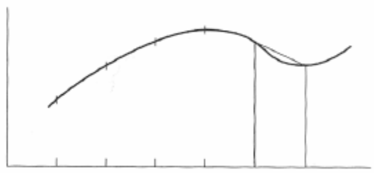


Trapezoid Rule

$$
\begin{aligned}
u_3(kT) &= u_3(kT - T) + \frac{T}{2}[-au_3(kT - T) \\
&\qquad + ae(kT - T) - au_3(kT) + ae(kT)] \\
&= \frac{1 - (aT/2)}{1 + (aT/2)} u_3(kT - T) + \frac{aT/2}{1 + (aT/2)} [e_3(kT - T) + e_3(kT)]
\end{aligned}
$$

Discrete Equivalents via Numerical Integration

- Trapezoid Rule



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&= \frac{1 - (aT/2)}{1 + (aT/2)} u_3(kT - T) + \frac{aT/2}{1 + (aT/2)} [e_3(kT - T) + e_3(kT)]
\end{aligned}
$$

- Transfer Function using this:

$$
\begin{aligned}
H_T(z) &= \frac{aT(z + 1)}{(2 + aT)z + aT - 2} \\
&= \frac{a}{(2/T)[(z - 1)/(z + 1)] + a}
\end{aligned}
$$

Trapezoid Rule

Tustin's Rule

Bilinear Transformation

Discrete Equivalents via Numerical Integration

Method	Approximation	
Forward rule (Euler's rule)	$s \leftarrow \frac{z-1}{T}$	$z = 1 + Ts$
Backward Rule	$s \leftarrow \frac{z-1}{Tz}$	$z = \frac{1}{1 - Ts}$
Trapezoid Rule (Tustin's rule, Bilinear transformation)	$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$	$z = \frac{1 + Ts/2}{1 - Ts/2}$

For Tustin's=Trapezoid:

$$\boxed{H_T(z) = H(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}}$$