Pole Placement via State Feedback

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Linear State Feedback for Continuous LTI Systems

- ***State feedback for pole placement***, in continuous time. Given the open loop, **time-invariant** continuous time state equation

$$\boxed{\begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned}}$$

"East coast" notation replaces F with A, G with B, H with C

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Apply the control law

$$\boxed{u(t) = Nr(t) - Kx(t)}$$

to obtain the closed loop state equation

$$\boxed{\begin{aligned}\dot{x}(t) &= (F - GK)x(t) + GNr(t)\\ y(t) &= Hx(t)\end{aligned}}$$

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$r = 0$ ***regulator problem***--make y(t) go to zero for t large

$r(t) \neq 0$ ***tracking problem*** [if r(t) is not constant, ***servomechanism problem***]

•If the system is **controllable**, by choosing K we can put the eigenvalues of (F-GK) ***anywhere we want***

Can place all the poles!

Finding the state feedback law, *K*

Note that for *r=0*

$$u = -Kx = -[K_1 \quad K_2 \quad \cdots \quad K_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Thus

$$\dot{x} = Fx - GKx$$

hence

$$\det[sI - (F - GK)] = 0$$

Select n desired closed-loop pole locations $s_1, \ldots, s_n$

Then the *desired* CL characteristic equation is

$$\boxed{\alpha_c(s) = (s - s_1)(s - s_2)\ldots(s - s_n) = 0}$$

Can we do this?

- Yes, if (F,G) is controllable – using "Ackermann's Formula" [details later]

- Numerical solution using MATLAB

  acker.m for SISO systems, n ≤ 10

  place.m works for MIMO case, higher orders, but no repeated poles in desired characteristic equation

  K = acker(F,G,p) *p is vector of n desired*

  K = place(F,G,p) *CL pole locations*

State Feedback for Pole Placement in Discrete Time

For system
$$\boxed{\begin{aligned}\dot{x}(t) &= Fx(t) + Gu(t)\\ y(t) &= Hx(t) + Ju(t)\end{aligned}}$$

$$\boxed{\begin{aligned}x[k+1] &= \Phi x[k] + \Gamma u[k]\\ y[k] &= Hx[k] + Ju[k]\end{aligned}}$$

State Feedback for Pole Placement in Discrete Time

Using state feedback $u = -Kx = -[K_1 \quad K_2 \quad \cdots \quad K_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ we have

$$x[k+1] = \Phi x[k] - \Gamma K x[k]$$

$$(zI - \Phi + \Gamma K)X(z) = 0$$

Closed Loop characteristic equation

$$|zI - \Phi + \Gamma K| = 0$$

We need to find the **n elements of K** to make the poles be in desired locations – let desired poles be $\alpha_1, \alpha_2, \ldots \alpha_n$

Desired characteristic eq
$$\boxed{\alpha_c(z) = (z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)}$$

State Feedback for Pole Placement in Discrete Time

Once again, using *Ackermann's formula* (**acker.m** or **place.m**)

In the continuous time case, we needed *controllability* to do this—this was the ability to drive any state to zero—or equivalently, the ability to get to any state from zero—or, equivalently, to get from any particular state value to any other state value

In discrete time, the relevant definitions are a bit more complicated.

State Feedback for Pole Placement in Discrete Time

Let's first consider the case of a single input and a system that is in control canonical form.

Suppose that the system is in **controllable** canonical form, and that the system has a single scalar input *u:*

$$x[k+1]=\underbrace{\begin{bmatrix}-a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}}_{\Phi}x[k]+\underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{\Gamma}u[k]$$

$$y[k]=\underbrace{\begin{bmatrix}b_1 & b_2 & \cdots & b_n\end{bmatrix}}_{H}x[k]$$

where the **characteristic polynomial of open loop system** is

$$\boxed{a(z)=(z-a_1)(z-a_2)\ldots(z-a_n)=z^n+a_1z^{n-1}+\ldots+a_n}$$

State Feedback for Pole Placement

Open Loop $\boxed{a(z) = (z - a_1)(z - a_2)\ldots(z - a_n) = z^n + a_1 z^{n-1} + \ldots + a_n}$

Desired Closed Loop $\boxed{\alpha_c(z) = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)}$

(want to find K to get this)

State Feedback Law $u[k] = -Kx[k]$

Closed Loop System $x[k+1] = \Phi x[k] - \Gamma K x[k]$

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Since

$$x[k+1] = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u[k]$$

State Feedback for Pole Placement

$$\boxed{\Phi - \Gamma K = \begin{bmatrix} -a_1 - K_1 & \ldots & -a_{n-1} - K_{n-1} & -a_n - K_n \\ 1 & \ldots & 0 & 0 \\ . & . & . & . \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

$$\boxed{\begin{aligned} K_i &= \alpha_i - a_i \\ K &= [\alpha_1 - a_1, \alpha_2 - a_2, \ldots, \alpha_n - a_n] \end{aligned}}$$

So when the system is in control canonical form, it is easy to find the desired state feedback gain *K* (for scalar u)

State Feedback for Pole Placement

$$K_i = \alpha_i - a_i \qquad K = [\alpha_1 - a_1, \alpha_2 - a_2, \ldots, \alpha_n - a_n]$$

when the system is in control canonical form, it is easy to find the desired state feedback gain K (for scalar u). If not,

Basic approach

- do mappings to turn the systems state space representation into control canonical form, then solve for K
- **Then get K back into the original state space coordinates**
- **This will give us Ackermann's formula.**
- **It will require controllability of the system's state space representation (so that we can get into control canonical form)**

Controllability

- A control system is said to be completely state controllable if it is possible to transfer the system from any arbitrary initial state to any desired state

Observability

- If every initial state x(0) can be determined from the observation of y(kT) over a finite number of sampling periods.