ME2025 Digital Control

Discrete Time Signals

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Sampled Signals

- Converting continuous time signal into discrete time system is unambiguous—e.g., sample value every T [note that we are throwing away information]

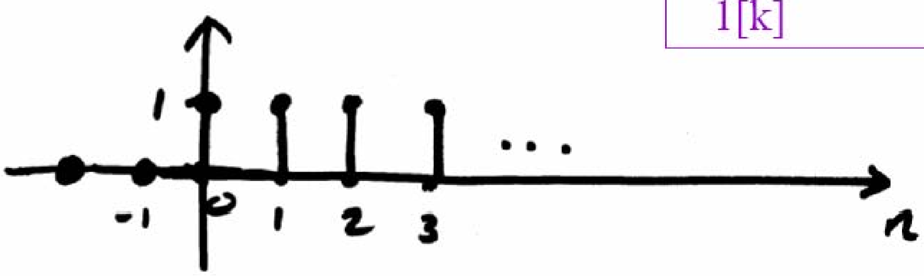
Given $f(t)$ to be a continuous time signal, $f(kT)$ is the value of $f(t)$ at $t = kT$. The discrete-time signal (or sequence) $f[k]$ is defined only for k an integer. So if we derive $f[k]$ from $f(t)$ by sampling every T seconds, where T is the sample period, we get:

$$f[k] = f(kT) = f(t)|_{t=kT}$$

Discrete Time Unit Step Function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

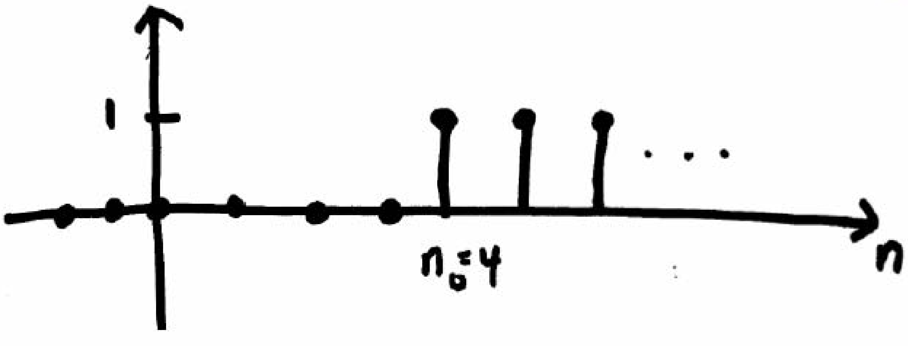
Alternate notation:
1[k]



Time-shifted unit step function

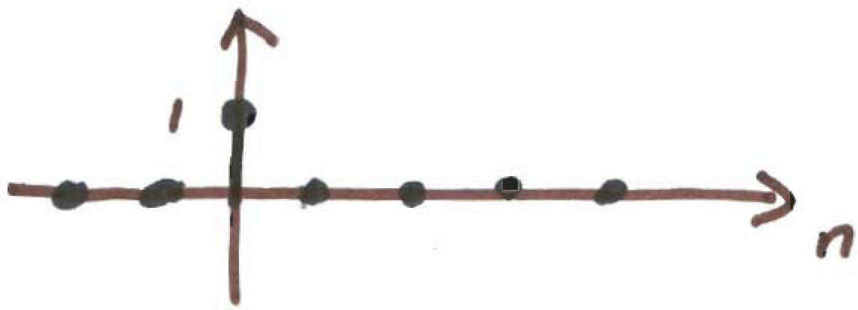
$$u[n - n_0] = \begin{cases} 1, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$$

Let n_0=4



Discrete Time Unit Impulse Function

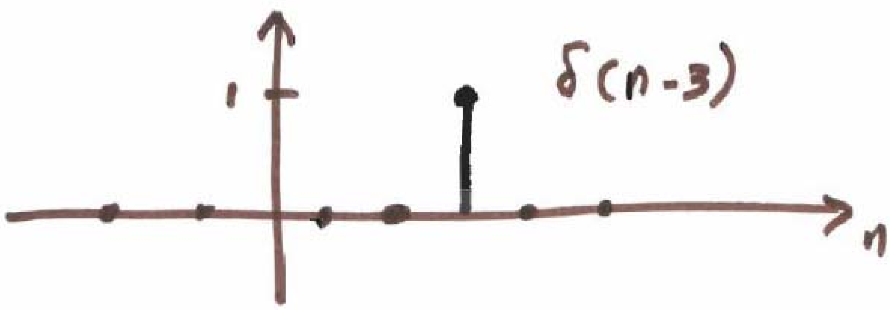
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Shifted Impulse Function

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

Let n_0=3



Comparison—DT and CT

Continuous time	Discrete time
$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$
$\delta(t) \equiv \frac{d}{dt} u(t)$	$\delta[n] = u[n] - u[n-1]$
$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$	$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
$\int_{-\infty}^{\infty} x(t)\delta(t-t_0) dt = x(t_0)$	$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$

Adding and Subtracting Signals

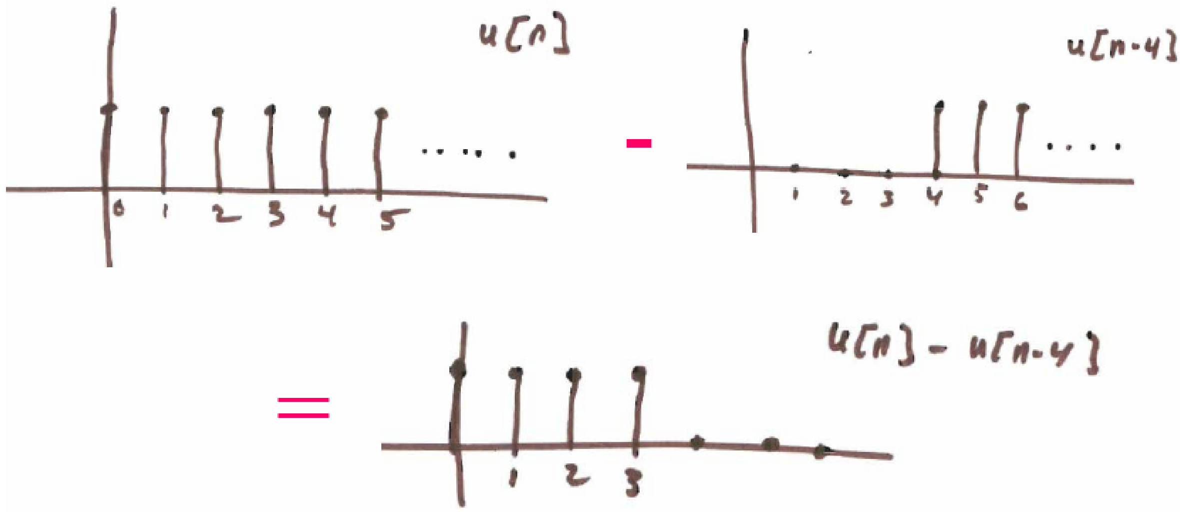
- Do it "point by point"
- Can do using a table, or graphically (or by computer program)
- Example:

$$x[n] = u[n] - u[n-4]$$

n	≤ -1	0	1	2	3	≥ 4
$x[n]$	0	1	1	1	1	0

Adding and Subtracting Signals

- Example: $x[n] = u[n] - u[n-4]$

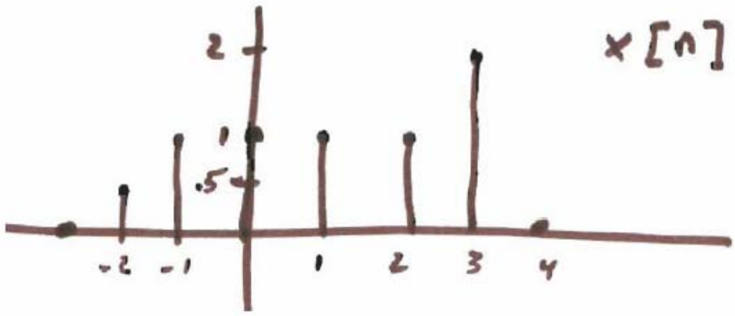


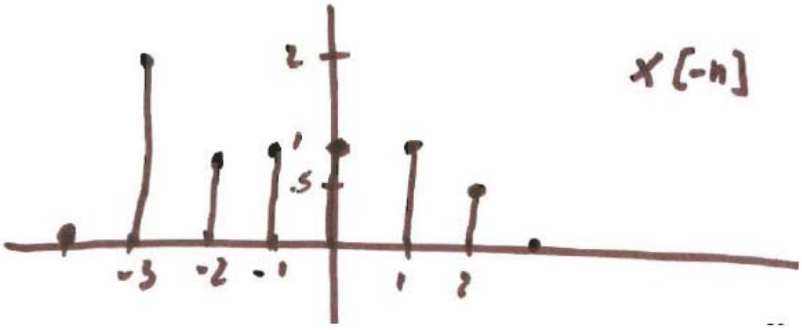
Time-Reversal of a Signal

$$y[n] = x[m]\big|_{m=-n} = x[-n]$$

This reversal operation precisely flips a signal about the vertical axis.

Time-Reversal of a Signal





Time-Scaling a Signal

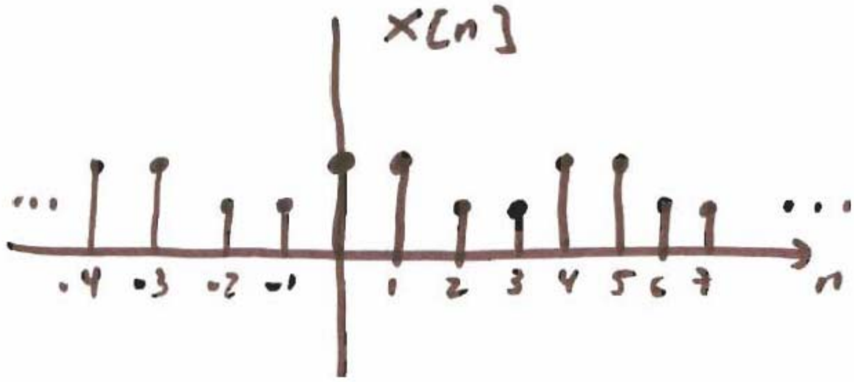
$$y[n] = x[m]\Big|_{m=an} = x[an]$$

If $|a| > 1$, then SPEED UP by a factor of a. If $|a| < 1$, then SLOW DOWN by a factor of a.

Unlike continuous time, there are **restrictions** on a!

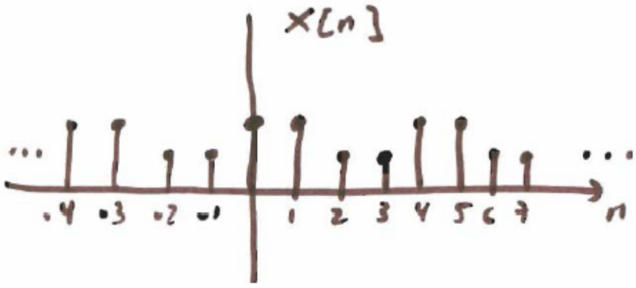
For **speeding up** (also known as "subsampling"), a must be an integer.

Time-Scaling a Signal-Subsampling

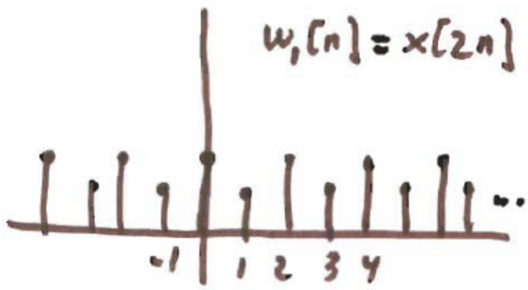


Find $w_1[n] = x[2n]$ and $w_2[n] = x[2n+1]$

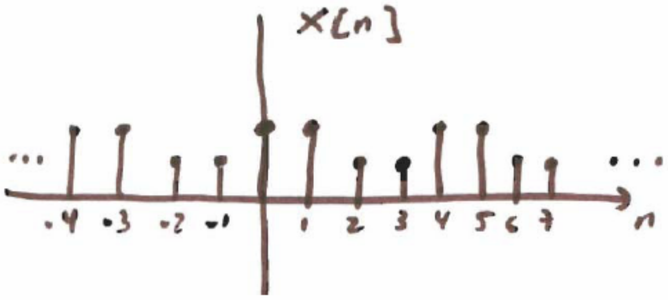
Time-Scaling a Signal-Subsampling

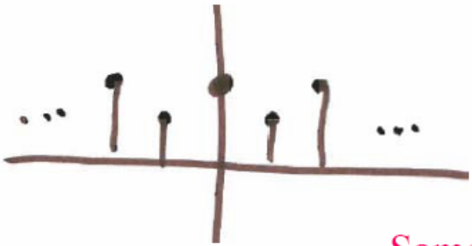


$w_1[n] = x[2n]$



Time-Scaling a Signal-Subsampling



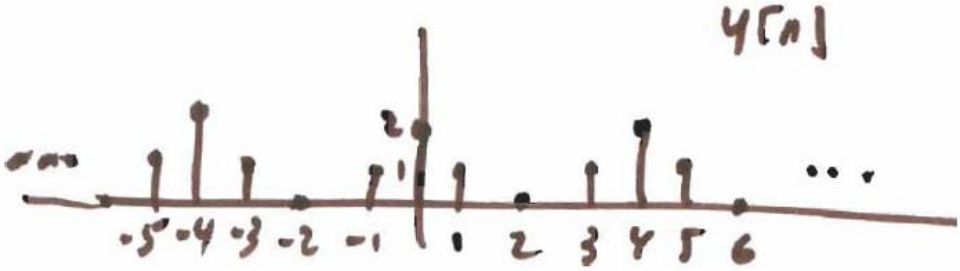


$w_2[n] = x[2n+1]$

Same in this case

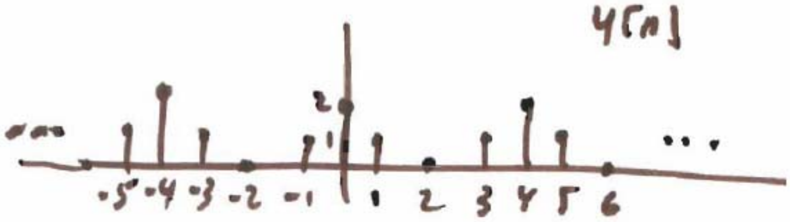
Time-Scaling a Signal-Subsampling

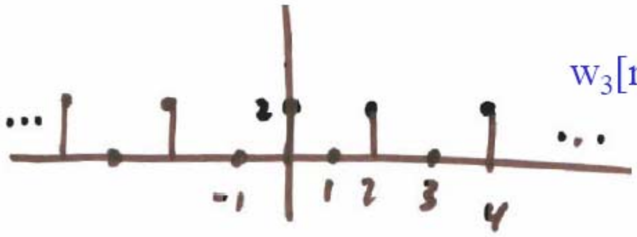
Try a different example…



Find $w_3[n] = y[2n]$ and $w_4[n] = y[2n+1]$

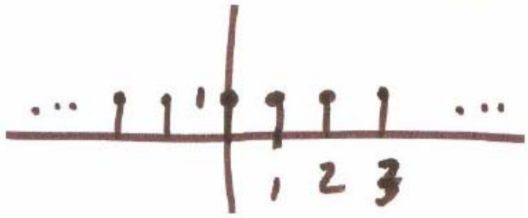
Time-Scaling a Signal-Subsampling





$w_3[n] = y[2n]$

$w_4[n] = y[2n+1]$



Time-Scaling a Signal-Slowing Down

For **slowing down** (expanding) a signal, you need $a = 1/K$ where K is an integer.

Example: Let $K = 2$ ($a = 1/2$) and find $z[n] = b[\frac{n}{2}]$

n	$z[n]$	$b[\frac{n}{2}]$
0	$z[0]$	$b[0]$
1	$z[1]$	??
2	$z[2]$	$b[1]$
3	$z[3]$	??

Values like $b[\frac{1}{2}]$ and $b[\frac{3}{2}]$ are not defined so how do we find $z[1]$ and $z[3]$??

Time-Scaling a Signal-Slowing Down

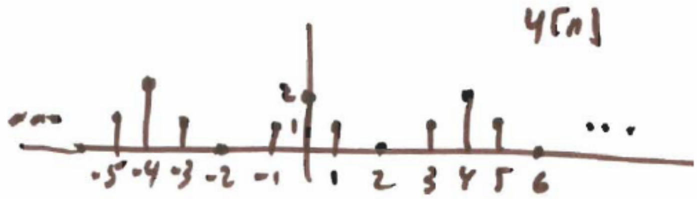
One solution is to INTERPOLATE
A simple, **but sub-optimal** interpolation rule is linear interpolation

n	$z[n]$	$b[\frac{n}{2}]$
0	$z[0]$	$b[0]$
1	$z[1]$	??
2	$z[2]$	$b[1]$
3	$z[3]$	??

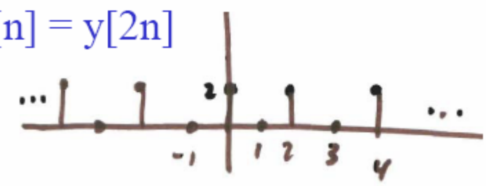
$$z[n] = \begin{cases} b[n/2], & n \text{ even} \\ 1/2\{b[(n-1)/2] + b[(n+1)/2]\}, & n \text{ odd} \end{cases}$$

Interpolation can be used in a simple compression scheme – just transmit every other sample and fill in missing the values at the receiver.

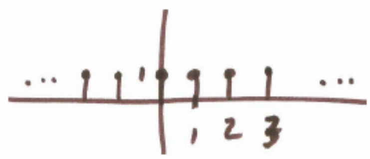
Recall earlier example…



$w_3[n] = y[2n]$



$w_4[n] = y[2n+1]$

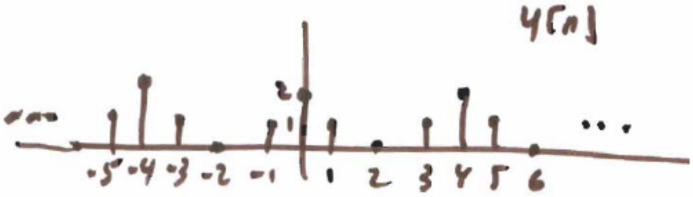


What does $z_3[n] = w_3[n/2]$ look like? (compute from w_3, using table and **linear interpolation**)

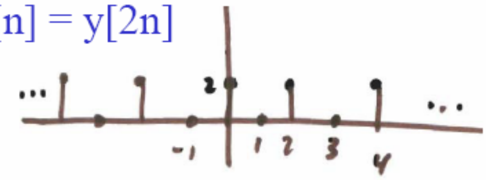
Looks just like y[n] !

Would look different if different interpolation

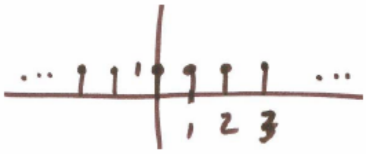
Recall earlier example…



$w_3[n] = y[2n]$



$w_4[n] = y[2n+1]$

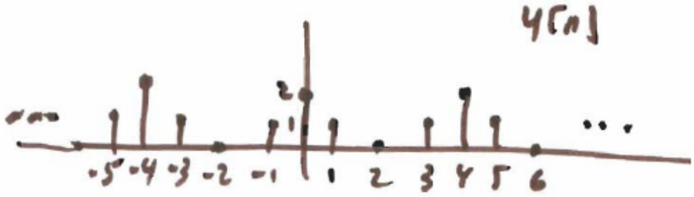


What does $z_4[n] = w_4[n/2]$ look like? (compute from w_4, using table and linear interpolation)

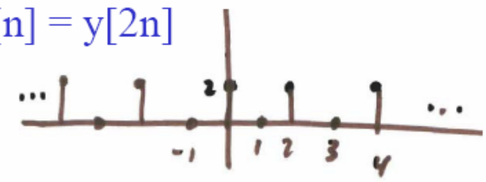
Would look different if different interpolation

Looks just like $w_4[n]$!

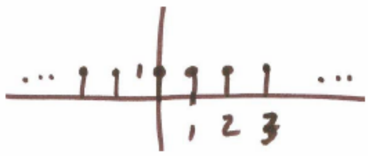
Recall earlier example…



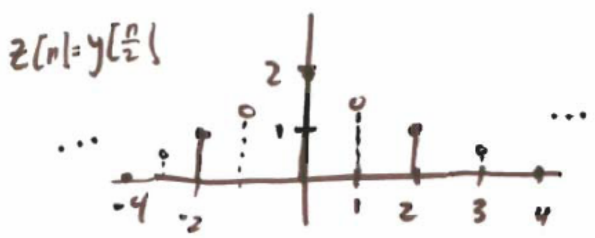
$w_3[n] = y[2n]$



$w_4[n] = y[2n+1]$

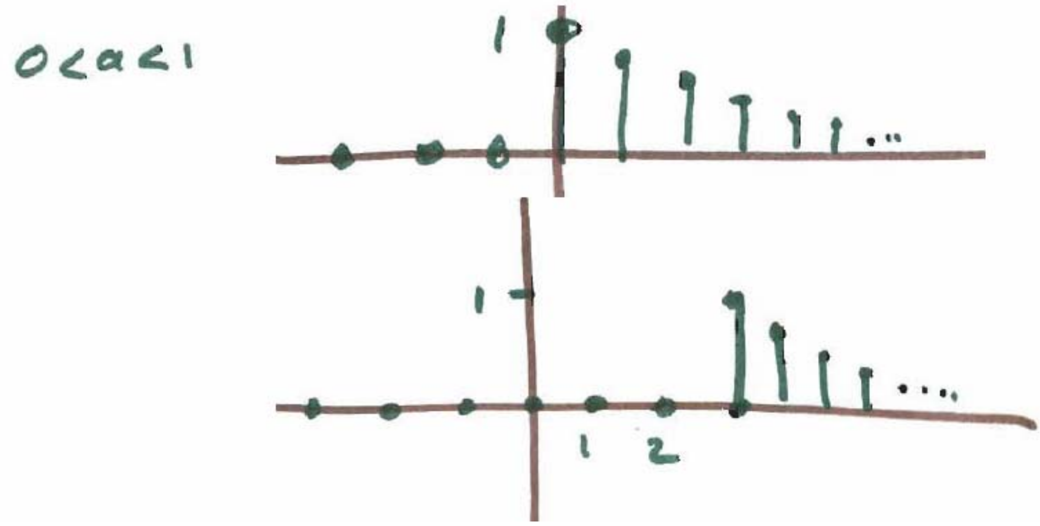


What does $y[n/2]$ look like, using linear interpolation?



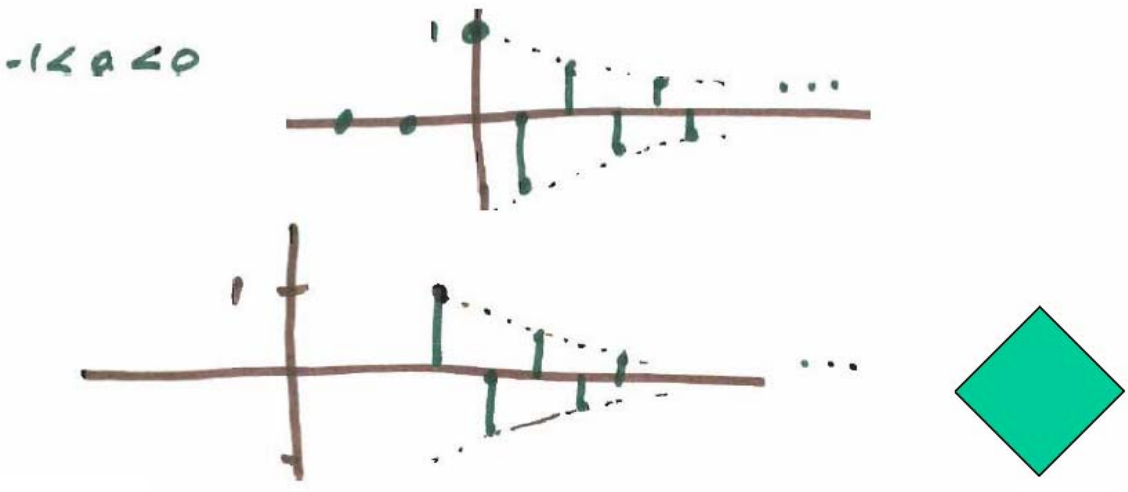
More Time-Shifting

Ex.: Given $x[n] = a^n u[n]$, $|a| < 1$, find and plot $y[n] = x[n-3]$



More Time-Shifting

Ex.: Given $x[n] = a^n u[n]$, $|a| < 1$, find and plot $y[n] = x[n-3]$



Combining Time Shifting and Scaling

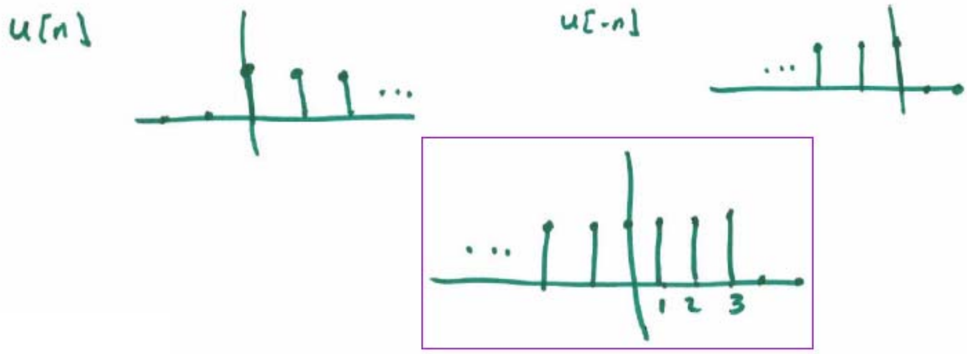
Ex. Find $u[3-n]$,

There are two direct ways to solve this example:

Method 1. Reverse (time scale of -1) then shift (delay) in time ($x[a(n+\frac{b}{a})]$):

$$z[n] = u[-n]$$

$$y[n] = z[n-3] = u[-(n-3)] = u[-n+3]$$



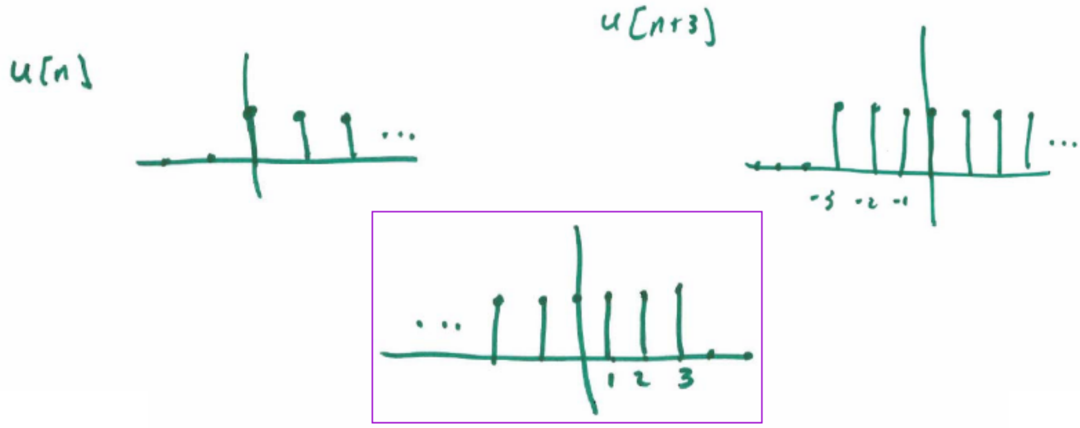
Combining Time Shifting and Scaling

Ex. Find $u[3-n]$,

Method 2. Advance in time then reverse ($x[an+b]$):

$$w[n] = u[n+3]$$

$$y[n] = w[-n] = u[-n+3]$$



Combining Time Shifting and Scaling

Be careful—for some cases method 1 doesn't work!
For example, if you want to form

$$z[n] = x[3-2n] = x\left[-2\left(n-\frac{3}{2}\right)\right],$$

What does it mean to shift a signal by 3/2 ?? To make sure, plug values into the table to check:

For other cases, method 2 doesn't work

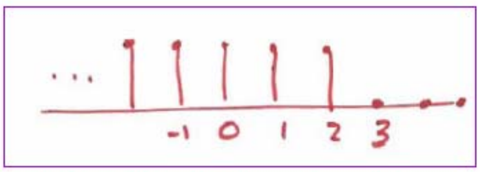
n	$z[n]$	$x[3-2n]$
0	$z[0]$	$x[3]$
1	$z[1]$	$x[1]$
2	$z[2]$	$x[-1]$
−1	$z[-1]$	$x[5]$
−2	$z[-2]$	$x[7]$

Combining Time Shifting and Scaling

Ex. Let $x[n] = 2u[n+2]$. Find $z[n] = x[3-2n]$.

x[n]=2u[n+2]

z[n]=x[3-2n]



Get this from table, using x[n] values. Note method 1 problem described before this example.

n	x[n]	3-2n	x[3-2n]
-4	0	11	2
-3	0	9	2
-2	2	7	2
-1	2	5	2
0	2	3	2
1	2	1	2
2	2	-1	2
3	2	-3	0
4	2	-5	0

Combining Time Shifting and Scaling

<u>Ex.</u>
Find $y[n] = x[2-2n]$:

$$x[2-2n] = x[-2(n-1)]$$

For method 1 approach, $v[n] = x[-2n]$, then delay by 1. Or, just plug in values of n in a table.

<u>Ex.</u> Let $y[n] = a^n u[n]$, where $a > 1$. Find and plot $z[n] = y[-2n+2]$.

Combining Time Shifting and Scaling

<u>Ex.</u> Let $y[n] = a^n u[n]$, where $a > 1$. Find and plot $z[n] = y[-2n+2]$.

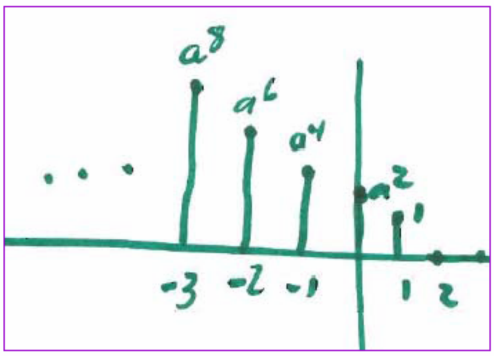
$z[0] = y[-2n+2] = y[2] = a^2$

$z[1] = y[-2\cdot 1+2] = y[0] = 1$

$z[2] = y[-2\cdot 2+2] = y[-2] = 0$

$z[-1] = y[-2(-1)+2] = y[4] = a^4$

$z[-2] = y[-2(-2)+2] = y[6] = a^6$

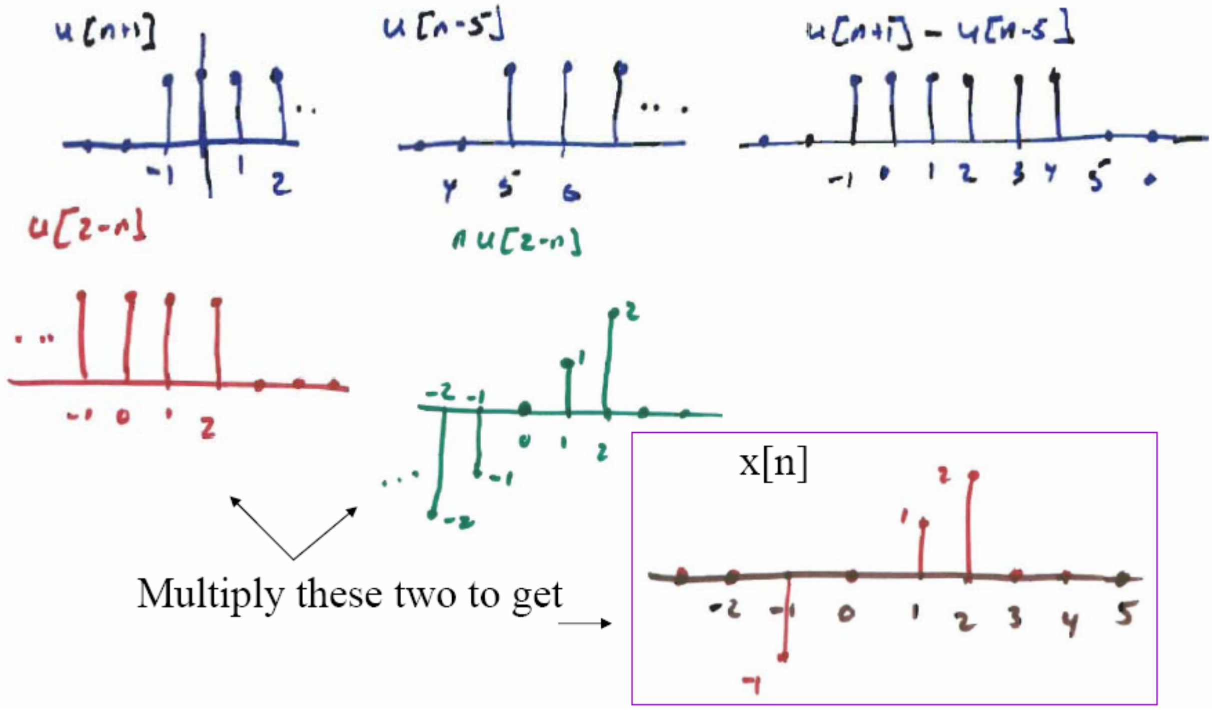


Amplitude Scaling

- Do point by point

Example: Find $x[n] = (u[n+1] - u[n-5])(nu[2-n])$

Example: Find $x[n] = (u[n+1] - u[n-5])(nu[2-n])$



Even and Odd Signals

Any discrete-time signal can be expressed as the sum of an **even signal** and an **odd signal**

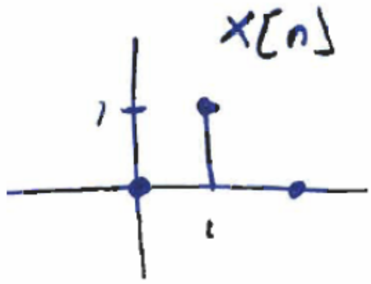
$$x[n] = x_e[n] + x_o[n]$$

$$\text{Even}: x_e[n] = x_e[-n]$$
$$\text{Odd}: x_o[n] = -x_o[-n]$$

$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$$
$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$$
$$x[n] = x_e[n] + x_o[n]$$

Even and Odd Signals

Ex. Given $x[n]$, find $x_e[n]$ and $x_o[n]$.



$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$$
$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$$
$$x[n] = x_e[n] + x_o[n]$$