Discrete Time Convolution

Jee-Hwan Ryu

School of Mechanical Engineering
Korea University of Technology and Education

Convolution

- An operation between two signals, resulting in a third signal
- Recall: in continuous time, convolution of two signals involves integrating the product of the two signals– where one of the signals is “flipped and shifted”
  - It doesn't matter which signal is flipped and shifted
  - Have to take care to get limits of integration correct
- In discrete time, convolution of two signals involves summing the product of the two signals – where one of the signals is “flipped and shifted”
  - It doesn't matter which signal is flipped and shifted
  - Have to take care to get limits of sum correct
- Convolution: an operation between the input signal to a system, and its impulse response, resulting in the output signal

Impulse Representation of Discrete-Time Signals

We can describe any discrete-time sequence $x[n]$ as:

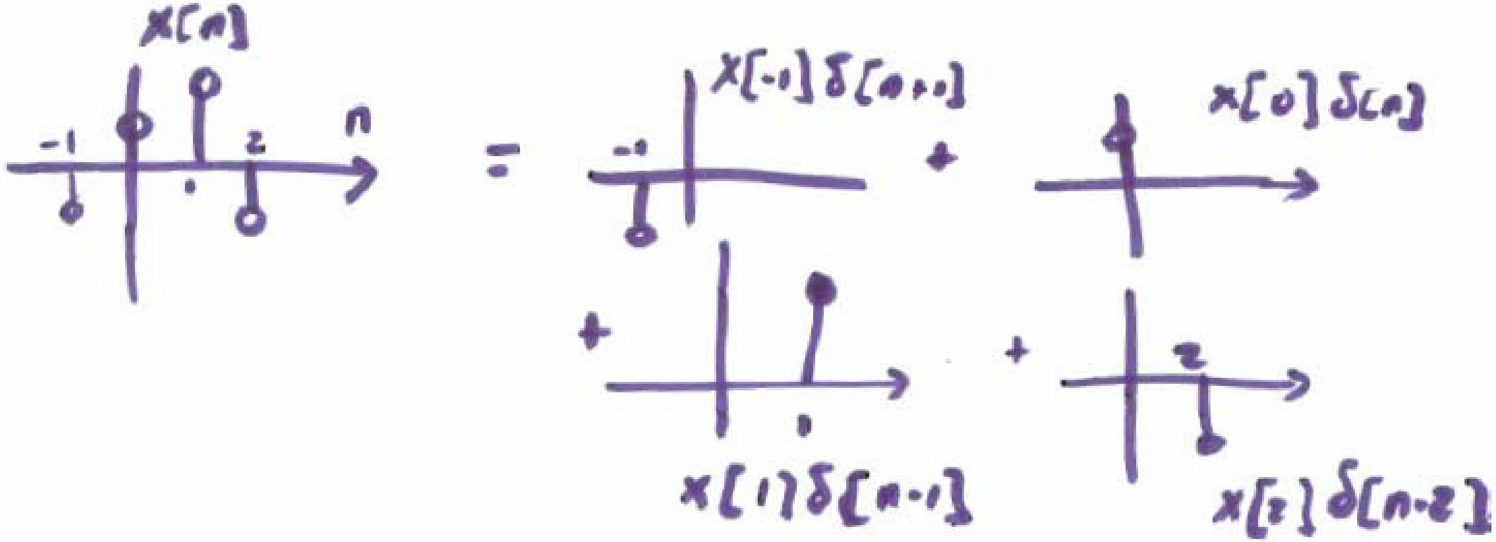
$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

which is equivalent to the more succinct notation

For each n, $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

like $$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

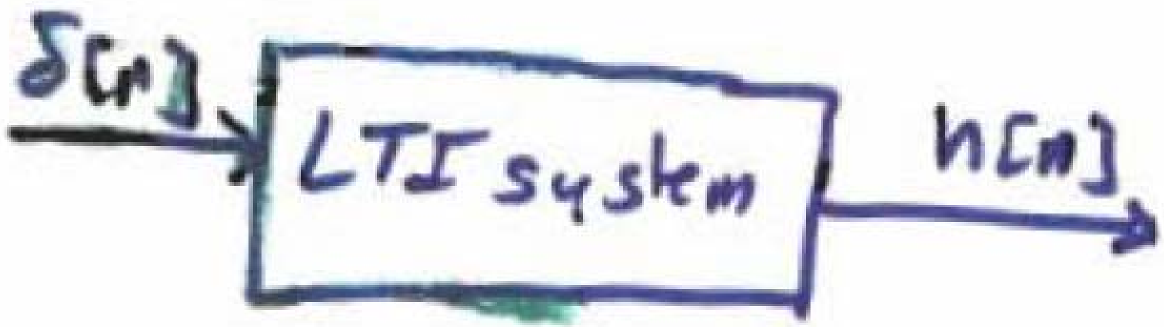
This equation expresses $x[n]$ as a series of impulse functions shifted in time, all scaled with weights $x[k]$. We will see this again when we show that the I/O relationship of a DT LTI system is a DT convolution.



The "impulse response" of a system is the output that results, in response to an input that is a unit impulse

Usually a system impulse response is denoted by $h_m[n]$ or, if the impulse response is time-invariant, by $h[n]$.

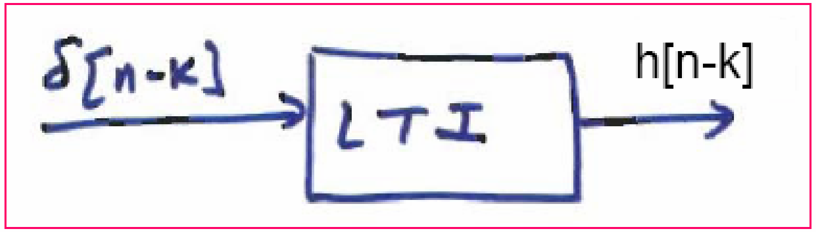
Impulse response of a time-varying system, to an impulse at time m



Using $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Representing the input signal as a sum of scaled, shifted impulses

and the fact that the system is linear plus knowledge that the response to $\delta[n-k]$ is some impulse response $h_k[n]$



Thus the output signal y[n], in response to input signal x[n] is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

Due to time-invariance, we get $h_k[n] = h[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \quad = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete Time Convolution

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The system's response properties (if zero initial conditions), are described by the impulse response sequence, *h[n]*

The Convolution Equation

Notation: * for convolution

Definition

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The output of an LTI system is the input, convolved with the impulse response (for LTID system)

Remember – the response of the system to an impulse (that is, it's impulse response) is a sequence of values that may go on forever

For the operation of convolution – both the input signal and the system impulse response are just sequences of Numbers – and the convolution operation * tells you how To compute the resulting sequence

You need to calculate *y[n]* for each time *n*, using

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

As in continuous-time, discrete-time convolution is commutative:

Let $m = n - k$, same as $k = n - m$, in above equation,

$$\sum_{n-m=-\infty}^{\infty} x[n-m]h[m] = \sum_{-m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=+\infty}^{-\infty} h[m]x[n-m] =$$

N assumed to be finite

$$\sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n]*x[n] = x[n]*h[n]$$

It doesn't matter whether the input signal or the impulse response gets flipped.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Steps to perform convolution:

1. Time reverse $h[k]$ and shift by n to form $h[n-k]$ (flip and shift)
2. Rewrite $x[n]$ as $x[k]$
3. Multiply $x[k]$ and $h[n-k]$ for all values of k.
4. Sum up $x[k]h[n-k]$ over all k to get $y[n]$
5. Do for all values of n .

HELPFUL HINTS:

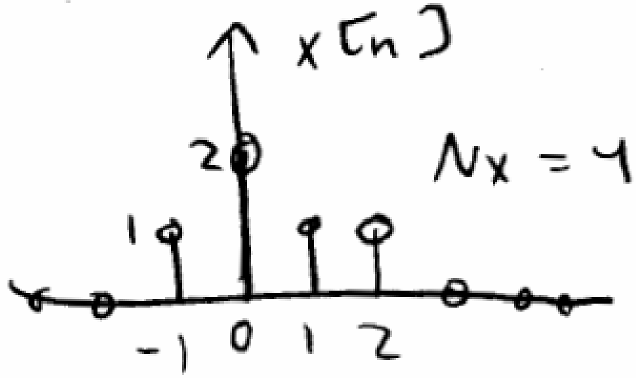
1. FLIP THE EASIER FUNCTION

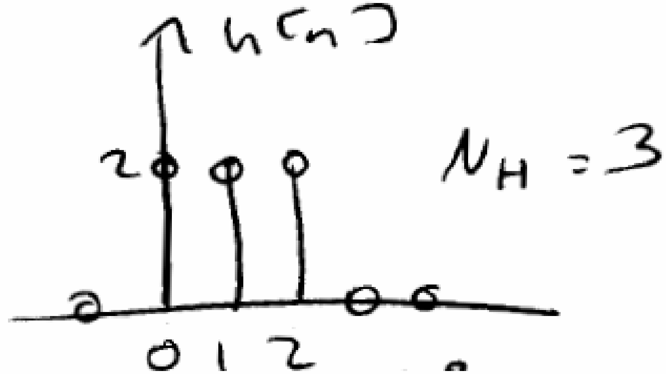
2. DRAW A PICTURE

Ex.

Find $x[n]*h[n] = y[n]$ where $x[n] = u[n+1] - u[n-3] + \delta[n]$ and $h[n] = 2(u[n] - u[n-3])$.

Plot x[n] and h[n]

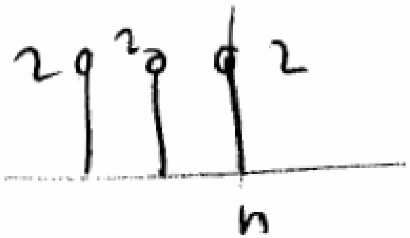


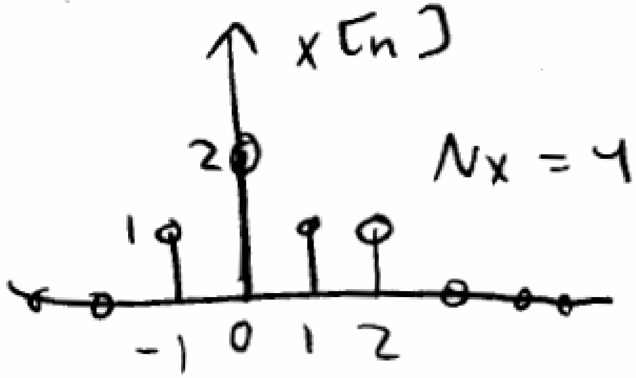


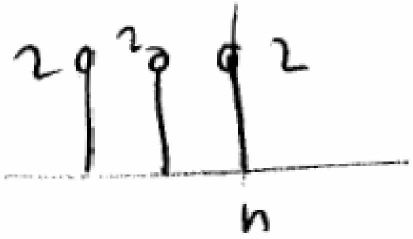
Then calculate for each n:

Flip the "easier" one

$$y[n] = x[n]*h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$







Then calculate for each *n*:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

For *n* < -1, all of the products in the sum are zero – so the sum is zero – so *y[n]* = 0

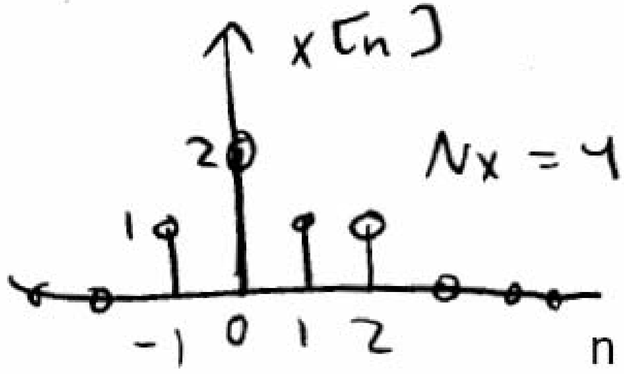
For *n* = -1, only one term in the sum is non-zero; it is (2)(1) = 2 so y[-1]=2

For *n* = 0, two terms in the sum are non-zero;

(2)(1) = 2

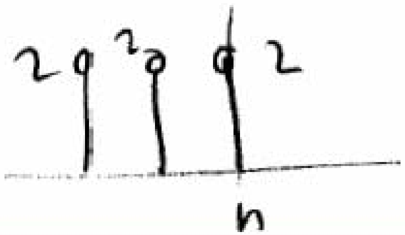
(2)(2) = 4

So y[0] = 2 + 4 = 6

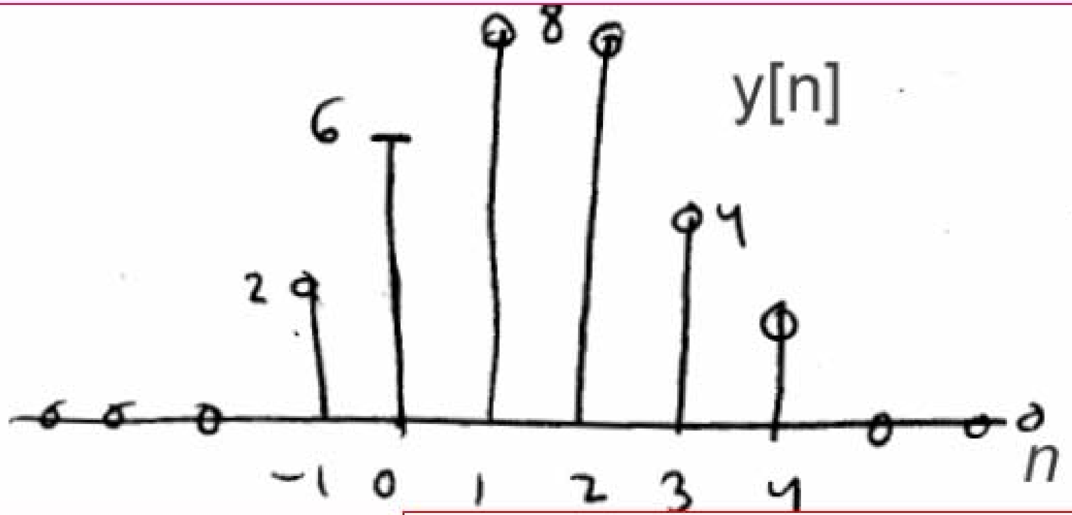


Continue this process.

What is largest value of *n* where there are any non-zero products?



$n = 4$



What is the "length" of y? (called N_y)

$N_Y = 6 = N_x + N_H - 1$

Note: $N_y = N_x + N_h - 1$, where N_i is the nonzero length of $i[n]$.

What is $x[n] * \delta[n]$?

$= x[n]$

The impulse function is the "identity" in the set of signals—that is, any signal multiplied by the impulse function is itself

<u>Ex.</u> Find $x[n] * \delta[n-n_0] \Rightarrow$

This is CONVOLUTION WITH DISCRETE-TIME IMPULSE $\Rightarrow$ Result is: Convolving a function with an impulse shifts the function to where the impulse is located.

$$\sum_{k=-\infty}^{\infty} x[n-k]\,\delta[k-n_0] = x[n-n_0]$$

(sifting property of the discrete time impulse)

Ex.

Find $y[n] = x[n] * h[n]$ where $x[n] = u[n] - u[n-3]$ and $h[n] = u[n] - u[n-5]$.

First, let's get the length of the signal using

$$N_y = N_x + N_h - 1$$

$N_X = 3$

$N_h = 5$

$N_Y = 3+5 - 1 = 7$

Ex.

Find $y[n] = x[n] * h[n]$ where $x[n] = u[n] - u[n-3]$ and $h[n] = u[n] - u[n-5]$.

Next, where does the signal "stop" and "start" ?

The (length 3) x signal starts at n=0 and ends at n=2 (zero for n > 2)
The (length 5) h signal starts at n = 0 and ends at n=4 (zero for n >4)

So the (length 7) signal y must start at n=0 and stay nonzero through n=6
It will ramp upwards (with slope 1), then maintain a constant value of 3
Then decrease back to zero (with slope 1)

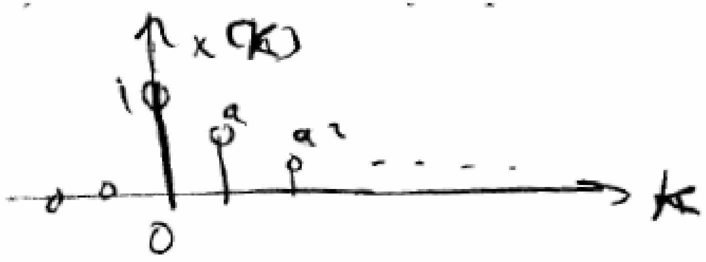
It's easiest to see this by flipping x and then shifting it

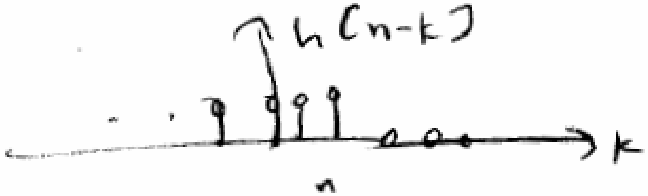
- For n < 0 y=0
- For n = 0, y = 1
- For n = 1, y=2
- For n = 2, y=3
- For n = 3, 4 y = 3
- For n = 5, y=2
- For n = 6 y=1
- For n > 6, y = 0

Ex. Find $y[n] = x[n] * h[n]$ where $x[n] = a^n u[n]$ and $h[n] = u[n]$.

$|a| < 1$

Try it both ways (first flip $x[n]$ and do the convolution and then flip $h[n]$ and do the convolution). Which method do you prefer?





$n < 0, \; 0$

$n \geq 0 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$

You know this sum

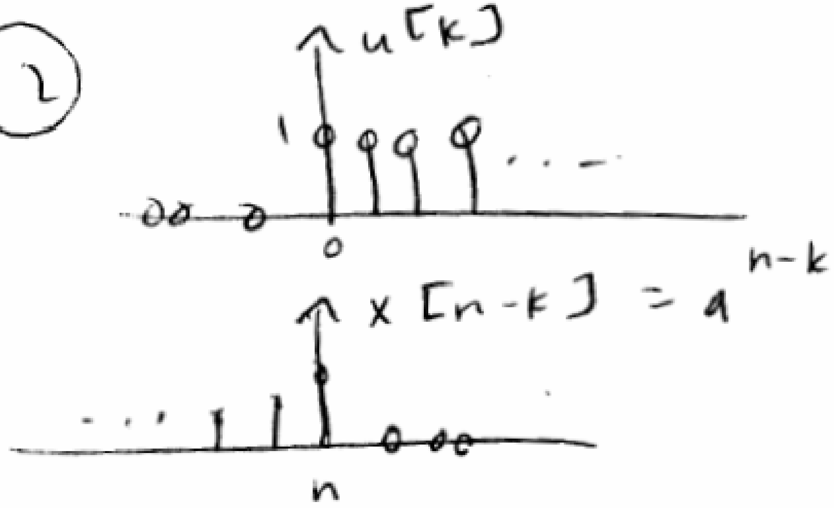
$y[n] = \left(\frac{1-a^{n+1}}{1-a} \right) u[n]$

Ex. Find $y[n] = x[n] * h[n]$ where $x[n] = a^n u[n]$ and $h[n] = u[n]$. $|a| < 1$

Try it both ways (first flip $x[n]$ and do the convolution and then flip $h[n]$ and do the convolution). Which method do you prefer?

Factor out to get into form you know

2



$n < 0$, 0

$n \geq 0$

$$\sum_{k=0}^{n} a^{n-k} = a^n \sum_{k=0}^{n} \left(\frac{1}{a}\right)^k$$

$$= a^n \left[\frac{1 - \left(\frac{1}{a}\right)^{n+1}}{1 - \frac{1}{a}} \right] =$$

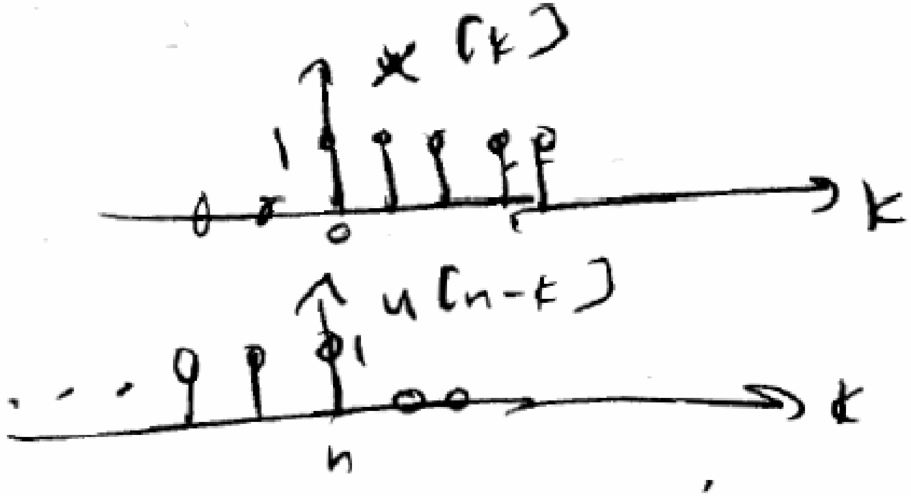
$$\frac{a^n - \frac{1}{a}}{1 - \frac{1}{a}} = \frac{a^{n+1} - 1}{a - 1} = \frac{1 - a^{n}}{1 - a}$$

$$\boxed{\therefore \left(\frac{1 - a^{n+1}}{1 - a} \right) u[n]}$$

More on DT convolution

<u>Ex.</u> $x[n] = h[n] = u[n]$. Find $y[n] = x[n] * h[n]$.

1. Do it graphically:



$n<0$, 0 $\quad$ $n \geq 0$: $\sum_{k=0}^{n} 1 = (n+1)$

$\therefore \ (n+1)u[n]$

2. Use convolution equation

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

$$\Rightarrow \sum_{k=0}^{\infty} u[n-k] \text{ since } u[k]=0,\ k<0$$

Limits to on sum indices come from where the signals are zero

Now,

$$u[n-k]=0, n-k<0 \text{ or } k>n \Rightarrow y[n]=\sum_{k=0}^{n}(1)=n+1$$

BUT what values of n is this good for?

$$u[k]=0,\ k<0 \text{ and } u[n-k]=0,\ k>n$$

$\Rightarrow$ only good for $0<k\le n \Rightarrow n\ge 0$.

Complete solution: y[n]=(n+1)u[n] , like before

Ex.

$$x[n] = b^n u[n]$$

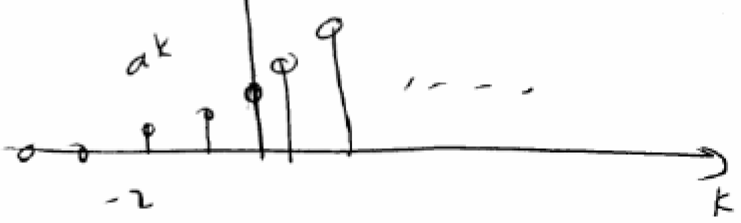
|a| < 1, |b| <1

$$h[n] = a^n u[n+2]$$

where $a \neq b$

Find $y[n] = x[n] * h[n]$.

$h[k] = a^k u[k+2]$



$|a| < 1$, $|b| < 1$

Value of $|a|$ and $|b|$ are issues if we have an infinite sum to evaluate

Flip x[k]

$n < -2$, 0

Factor out b^n, and get in form you know

$$n \geq -2 \qquad \sum_{k=-2}^{n} b^{n-k} a^k = b^n \sum_{k=-2}^{n} \left(\frac{a}{b}\right)^k =$$

let $\ell = k + 2$

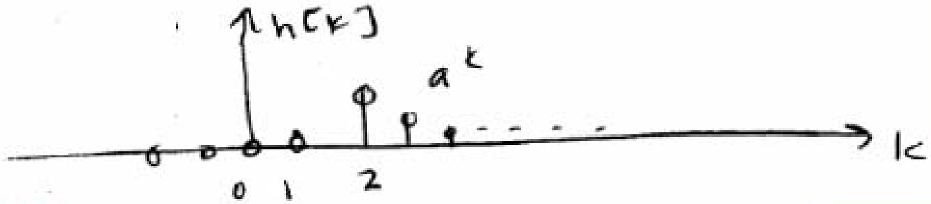
$$b^n \left(\frac{a}{b}\right)^{-2} \sum_{\ell=0}^{n+2} \left(\frac{a}{b}\right)^{\ell}$$

$$\boxed{y[n] = b^n \left(\frac{a}{b}\right)^{-2} \left(\frac{1 - \left(\frac{a}{b}\right)^{n+3}}{1 - \frac{a}{b}} \right) u[n+2]}$$

Ex. Compute output of a system with impulse response

$$h[n] = a^n u[n-2], \quad |a| < 1$$

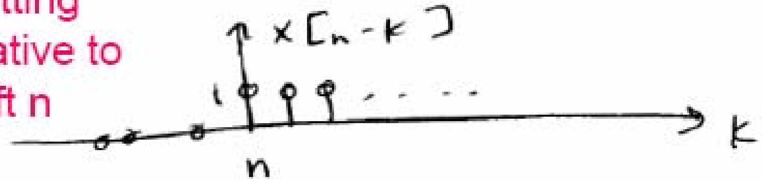
when the input is $x[n] = u[-n]$.



convolution sum

$$n \le 2 \quad y[n] = \sum_{k=2}^{\infty} a^k = \frac{a^2}{1-a}$$

Plotting relative to shift n



$$n > 2 \quad \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}$$

Flipping x[n] because it is simpler

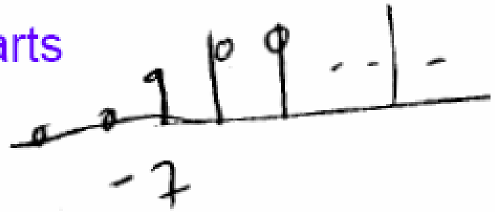
$$y[n] = \frac{a^2}{1-a} u[2-n] + \frac{a^n}{1-a} u[n-3]$$

A constant

Varies with n

$\boxed{\text{Ex.}}$ Given $x[n] = u[n]$ and $h[n] = a^n u[n+7]$, find $y[n] = x[n] * h[n]$

h[n] starts at -7



$$y[n] = \begin{cases} 0, & n < -7 \\ \displaystyle\sum_{k=-7}^{n} a^k, & n \geq -7 \end{cases}$$

From convolution sum

let $l = k + 7$

$$\sum_{l=0}^{n+7} a^{(l-7)} = a^{-7}\left[\frac{1 - a^{n+8}}{1-a}\right]$$

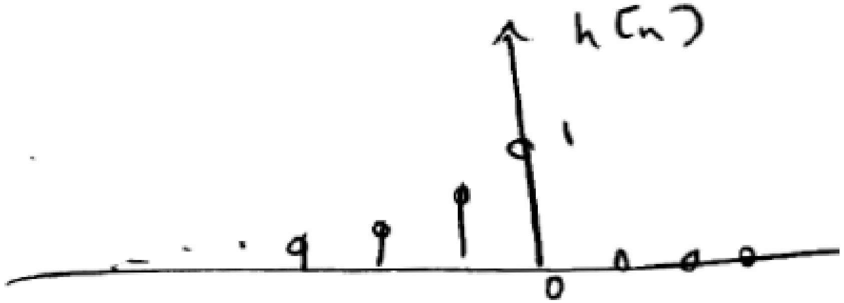
$$\boxed{\therefore\ a^{-7}\left[\frac{1-a^{n+8}}{1-a}\right] u[n+7]}$$

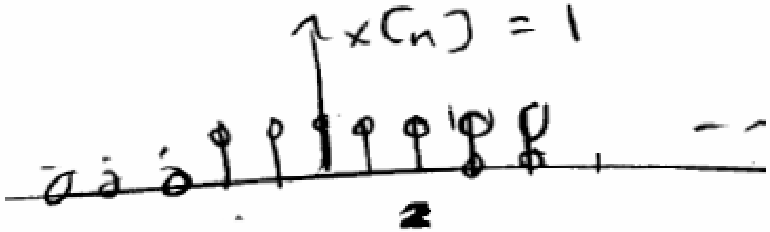
$\boxed{\text{Ex.}}$ Given $x[n] = u[-n+2]$ and $h[n] = a^n u[-n]$, find

$$y[n] = x[n] * h[n]$$

The output should be left-sided.

Both signals are left-sided





Flip h[n] in the convolution sum. We see that for n>2, y[n]=0.

$$n \le 2 \qquad \sum_{k=n}^{2} a^{n-k} = a^n \sum_{k=n}^{2} \left(\frac{1}{a}\right)^k$$

Upper limit in sum from x[n]=u{-n+2]. Lower limit in sum from h[n-k]=a$^{(n-k)}$u[k-n]]

$$\boxed{a^n \left(\frac{1}{a}\right)^n \sum_{k=0}^{2-n} \left(\frac{1}{a}\right)^k = \frac{1-\left(\frac{1}{a}\right)^{3-n}}{1-\frac{1}{a}}}$$

Here are some examples of discrete-time impulse responses:

Unit delay: $h[n] = \delta[n-1]$
Unit advance: $h[n] = \delta[n+1]$
Accumulator: $h[n] = u[n]$
Edge detector: $h[n] = \delta[n] - \delta[n-1]$

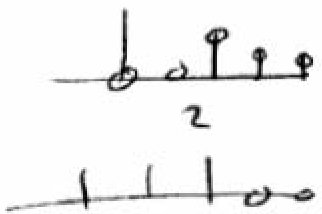
Step Response of a Discrete-Time System

The step response of an LTI system is just the response of the system to an input equal to unit step. We can denote this as $s[n]$.

$\boxed{\text{Ex.}}$ Compute DT step response of an LTI system with $h[n] = a^n u[n-2]$.

$$u[n] * a^n u[n-2] = s[n]$$

$\boxed{n < 2,\ 0}$



$$\boxed{n \geq 2 \qquad \sum_{k=2}^{n} a^k = a^2 \sum_{k=0}^{n-2} a^k = a^2\left(\frac{1-a^{n-1}}{1-a}\right)}$$

$$\boxed{s([n]= \quad a^2\left(\frac{1-a^{n-1}}{1-a}\right) u[n-2]}$$

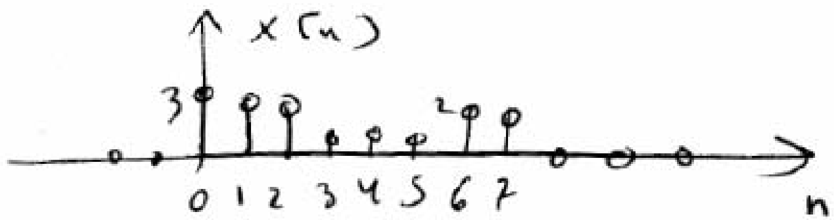
$\boxed{\text{Ex.}}$ Given $h[n] = a^n u[n]$ and $x[n] = 3u[n] - 2u[n-3] + u[n-6] - 2u[n-8]$, use SUPERPOSITION to find $y[n] = x[n] * h[n]$.

$|a| < 1$

Did this problem earlier--

$$s[n] = \left(\frac{1 - a^{n+1}}{1 - a} \right) u[n]$$

Using superposition (because the system is linear), we can obtain the desired signal as follows



$$y[n] = 3s[n] - 2s[n-3] + s[n-6] - 2s[n-8]$$

This "divide and conquer" method is very useful