

ME2025 Digital Control

The Z Transform

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Wanted: Discrete time equivalent of the Laplace Transform

Why?

- a way to solve difference equations
- a way to do frequency domain analysis and design of discrete time systems

Solution: The Z Transform

The Z Transform

Definition: For a discrete time sequence


$$x[k] = \dots, x[-2], x[-1], x[0], x[1], x[2], \dots$$

the **Z Transform** of $x[k]$ is defined to be

$$X(z) \equiv \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

where it is assumed that values of z exist such that the summation converges. Here z takes values in the Complex Plane.

The values of z where the sum exists are the Region of Convergence (ROC).

 If no such values of z exist, then $X(z)$ does not exist – that is, **the signal $x[k]$ does not have a Z-transform.**

Recall the Laplace Transform:

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt$$

The z-transform is the discrete time counterpart of the Laplace Transform:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n}$$

s and z take values in the complex plane

t and n are time variables

Infinite integral replaced by infinite sum

e^{-st} replaced by z^{-n}

The Z Transform

$$X(z) \equiv \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Used in Digital Control, Digital System Analysis, and to solve Linear, Constant Coefficient Difference Equations
---(like Laplace used to solve linear, constant coefficient Differential equations)
—transformed equations are algebraic

Definitions of z Transforms

For signal $h[n]$,

$$H(z) \equiv \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Sum goes
over all
integer
values

is the bilateral (2-sided) z-transform.

For example, $h[n]$ could be
the impulse response
sequence of a system

The bilateral Z-transform will work for all signals (one-sided, two sided, finite length, etc).


For Laplace Transforms we considered values of s where the infinite integral converges; here we consider values of z where the infinite sum converges (the "region of convergence")

Aside: You can relate the z-transform and Laplace transform directly when you are dealing with sampled signals:

Take a CT signal $f(t)$ and sample it:

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT)$$

Sampling = multiplying by an infinite series of time-shifted impulses, and then summing



Take a CT signal $f(t)$ and sample it:

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT)$$

The Laplace transform of the sampled signal is

$$\begin{aligned} L[f_s(t)] &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT) \right] e^{-st} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(nT) \delta(t - nT) e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-st} dt = \sum_{n=-\infty}^{\infty} f(nT) e^{-snT} \end{aligned}$$

by the sifting property.

Let $f[n] = f(nT)$ and $z = e^{sT}$, then

Thus the z-transform with $z=e^{sT}$ is the same as the Laplace transform of a sampled signal!

$$\begin{aligned} F(z) &= \sum_{n=-\infty}^{\infty} f[n] z^{-n} \\ F(z)|_{z=e^{sT}} &= \sum_{n=-\infty}^{\infty} f[n] e^{-sTn} \\ &= \sum_{n=-\infty}^{\infty} f(nT) e^{-snT} \\ &= L[f_s(t)] \end{aligned}$$

Unilateral (one-sided) Z Transform

We can also define a **unilateral** z-transform (one-sided) by

*For causal impulse
response sequence*

$$H(z) \equiv \sum_{n=0}^{\infty} h[n]z^{-n}$$

*Sum goes over
all non-negative
integer values*

One-sided Z Transform

- Key property—will be useful later, when considering initial conditions of difference equations:

$$\begin{aligned} Z (x [k + 1]) &= \sum_{n=0}^{\infty} x [n + 1] z^{-n} \\ &= \left(\sum_{n=0}^{\infty} x [n] z^{-(n-1)} \right) - x [0] z \\ &= z \sum_{n=0}^{\infty} x [n] z^{-n} - x [0] z \\ &= zX (z) - zx [0] \end{aligned}$$

Region of convergence (for bilateral Z Transform)

if you don't know the ROC, you can't determine $x[k]$ uniquely from $X(z)$

You must specify the ROC for the *bilateral* Z transform to uniquely determine the corresponding time domain function

From the Definition of the Z transform, we can write the Z transform of signal $x[n]$ as

$$X(z) = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$



This sum extends out to infinity in both directions if signal $x[n]$ does

Some Definitions

- $x[n]$ is right-sided if $x[n] = 0, n < n_0$
- $x[n]$ is left-sided if $x[n] = 0, n > n_0$

Region of convergence (for bilateral Z Transform)

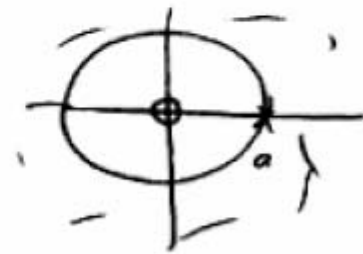
The *right-sided* discrete time signal

$$x_1[n] = a^n u[n] \quad \text{Here } |a| < 1, \text{ a real-valued}$$

has Z-transform

$$X_1(z) = \frac{z}{z - a}$$

with region of convergence $|z| > |a|$



All of plane
outside a circle
of radius $|a|$

Region of convergence (for bilateral Z Transform)

Next consider the *left-sided* discrete time signal

$$x_2[n] = -a^n u[-n-1]$$

$|a| < 1$, a real-valued

$$\begin{aligned} X_2(z) &= - \sum_{n=-\infty}^{-1} (az^{-1})^n = - \sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n = -\frac{\frac{z}{a}}{1 - \frac{z}{a}} \\ &= \frac{z}{z-a} \quad \left| \frac{z}{a} \right| < 1 \quad |z| < |a| \end{aligned}$$

There will be a left-sided and a right-sided time function that have the same z-transform, except for the region of convergence

You must specify the ROC for the *bilateral Z* transform to uniquely determine the corresponding time domain function

We've just seen that right-sided signals have an ROC of the form

$$|z| > r_{\max}$$

$$X(z) = \sum_{n=n_0}^{\infty} x[n]z^{-n}$$

For right-sided $x[n]$

$$X(z) = \sum_{n=n_0}^{\infty} x[n] \left(\frac{1}{z}\right)^n$$

As $n \rightarrow \infty$, need $(1/z)^n \rightarrow 0$ for sum to converge.

Happens for z OUTSIDE the poles $|z| > r_{\max}$.

Left-sided signals have an ROC of the form

$$|z| < r_{\min}$$

For left-sided $x[n]$
$$X(z) = \sum_{n=-\infty}^{n_0} x[n]z^{-n}$$

As $n \rightarrow -\infty$, need $(1/z)^n \rightarrow 0$ or $z^n \rightarrow 0$

Happens for z **INSIDE** the poles (rather than outside)

What about $z = \infty$?

If $x[n]$ is not causal but is still right-sided, e.g. $x[n] = u[n+1]$, then

$$X(z) = \sum_{n=-1}^{\infty} z^{-n} = z + \sum_{n=0}^{\infty} z^{-n}$$

will not converge at $z = \infty$, and we won't include it in the ROC.

Thus we can tell if a system is causal from the ROC of the Z-transform of its impulse response

$$|z| > r_{max} \Rightarrow \text{CAUSAL}$$

$$\infty > |z| > r_{max} \Rightarrow \text{right - sided but not causal}$$

What about convergence at $z=0$?

If $x[n]$ is left-sided but has values a some times >0 then we generally can't include 0 in the ROC

For example, for $x[n]=u[-n+1]$

$$X(z) \equiv \sum_{n=-\infty}^1 z^{-n} = z^{-1} + \sum_{n=0}^{\infty} z^n$$

does not converge at $z=0$. **So don't include 0 in the ROC.**

2-sided signals have ROC of the form

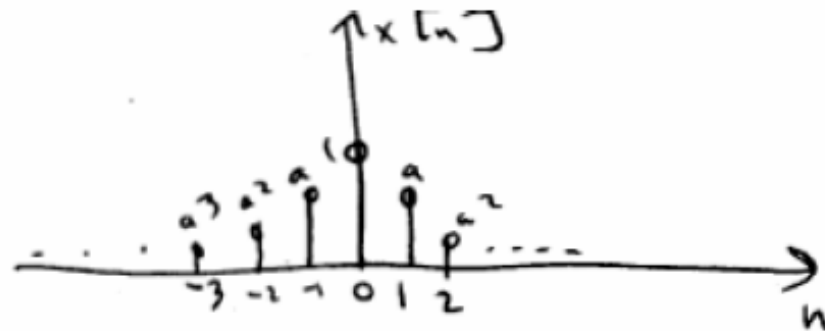
$$|a| < |z| < |b| \quad (\text{a "washer" or "donut"})$$

Finite duration signals have the whole plane as the ROC, except possibly excluding $z=0$ or ∞

$$\delta[n-1] \leftrightarrow z^{-1}, |z| > 0$$

$$\delta[n+1] \leftrightarrow z, |z| < \infty$$

Ex: Find the z Transform of $x[n] = a^{|n|}$ where $|a| < 1$



$$x[n] = a^{|n|}$$

$$|n| = \begin{cases} n, & n \geq 0 \\ -n, & n < 0 \end{cases}$$

Find its ROC

$$\therefore X(z) = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} =$$

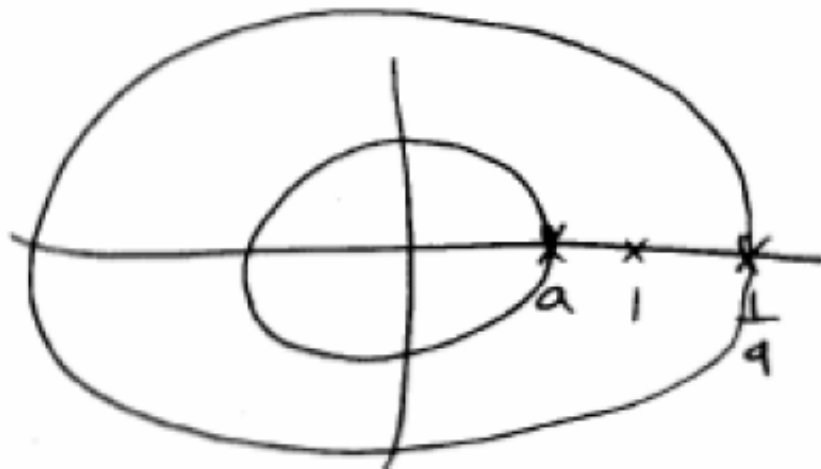
$$\sum_{n=1}^{\infty} (az)^n + \sum_{n=0}^{\infty} (az^{-1})^n = \frac{az}{1-az} + \frac{1}{1-az^{-1}}$$

ROC is intersection of that for each of these two poles

$$|az| < 1, \quad |az^{-1}| < 1$$

$$|z| < \frac{1}{|a|}, \quad |z| > |a|$$

$$\therefore X(z) = \frac{1-a^2}{(1-az)(1-az^{-1})} \quad |a| < |z| < \left|\frac{1}{a}\right|$$



Ex. Find the z-Transform of $x[n] = 3^n u[-n-1] + 4^n u[-n-1]$.

$$- \frac{z}{z-3}$$

$$|z| < 3$$

$$- \frac{z}{z-4}$$

$$|z| < 4$$



$$|z| < 3$$

On the other hand, for $x[n] = -3^n u[n] - 4^n u[n]$ we have ROC $|z| > 4$

Ex. Find the z-transform of $\frac{1}{2}\delta[n-1] + 3\delta[n+1]$.
What is its ROC?

A finite length signal

$$\sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\delta[n-1] + 3\delta[n+1] \right] z^{-n}$$
$$= \frac{1}{2}z^{-1} + 3z$$

So $0 < |z| < \infty$

is the ROC—the entire plane except the origin

Ex. Find the z-transform of $h[n] = (.5)^n u[n-1] + 3^n u[-n-1]$.

Would the system represented by this impulse response be BIBO stable?

$$X(z) = \sum_{n=1}^{\infty} (.5z^{-1})^n - \frac{z}{z-3}$$

$$= \frac{.5z^{-1}}{1-.5z^{-1}} - \frac{z}{z-3}$$

$|z| > .5$

$|z| < 3$

$= \frac{.5}{z-.5} - \frac{z}{z-3}, .5 < |z| < 3$



Ex. Find the z-transform of $x[n] = r^n \sin(bn)u[n]$ using inverse Euler's formula.

$$r^n \left[\frac{1}{2j} (e^{jbn} - e^{-jbn}) \right] u[n]$$

$$X(z) = \frac{1}{2j} \sum_{n=0}^{\infty} (e^{jb} z^{-1})^n - \frac{1}{2j} \sum_{n=0}^{\infty} (r e^{-jb} z^{-1})^n$$

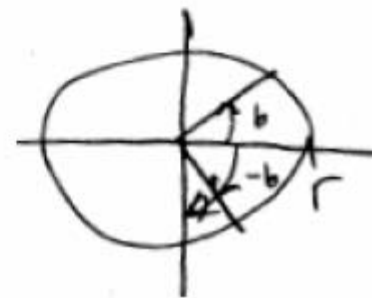
$$= \frac{1}{2j} \frac{1}{1 - r e^{jb} z^{-1}} - \frac{1}{2j} \frac{1}{1 - r e^{-jb} z^{-1}}$$

$$|r e^{-1} e^{jb}| < 1 \quad |r e^{-jb} z^{-1}| < 1$$

$$|z| > |r| \quad |z| > |r|$$

After a lot of algebra....

$$\frac{r z \sin b}{z^2 - 2r \cos b z + r^2}, \quad |z| > |r|$$



Transfer Function

For system impulse response (or other signal) $h[n]$, its z-Transform is often called its **Transfer Function**, $H(z)$.

Consider $H(z) = N(z)/D(z) = \text{Numerator/Denominator}$.

The values of z that make $D(z) = 0$ are the system **poles**.

The values of z that make $N(z) = 0$ are the system **zeros**.

Note: these are values of z – not values of z^{-1}

Characteristic equation



Properties of Z Transforms

Linearity

$$ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z)$$

where the new ROC $R' \supseteq R_x \cap R_y$.

That is, the operation of "taking the Z transform" is linear

Properties of Z Transforms

Time Shift (delay or advance)

$$x[n - n_0] \xleftrightarrow{Z} z^{-n_0} X(z)$$

where the new ROC is the same as R_x with the possible addition or deletion of the origin or infinity.


$$\sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} \quad \text{let } l = n - n_0$$
$$\sum_{l=-\infty}^{\infty} x[l] z^{-(l+n_0)} = z^{-n_0} \sum_{l=-\infty}^{\infty} x[l] z^{-l} = z^{-n_0} X(z)$$

Delay of d means the Z transform is multiplied by

$$z^{-d}$$

Convolution

$$y[n] = x[n] * h[n] \xleftrightarrow{z} X(z)H(z)$$

Convolution in time domain  multiplication in Z domain

where the new ROC $R_y \supseteq R_x \cap R_h$.

The intersection of the separate
signal ROCs
is contained in the ROC of $Y(z)$

Proof of Convolution Property

$$y[n] = x[n] * h[n] \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right] z^{-n}$$

Switching the order of the summations

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k] z^{-n}$$

Now, let $m = (n - k)$ and we get:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k] \left[\sum_{m=-\infty}^{\infty} h[m] z^{-(m+k)} \right] \\ &= \sum_{k=-\infty}^{\infty} x[k] z^{-k} \sum_{m=-\infty}^{\infty} h[m] z^{-m} \\ &= X(z)H(z) \end{aligned}$$

ROC of $Y(z)$ is the **intersection** of ROCs of X and H , except if pole cancels zero (then it can be bigger)

$$R'_y \supseteq R_x \cap R_h.$$

Ex.

$$X(z) = \frac{z-3}{z-2}, \quad |z| > 2$$

$$H(z) = \frac{z-2}{z-3}, \quad |z| > 3$$

$$R_X \cap R_H = |z| > 3$$

$$\text{But } Y(z) = H(z)X(z) = 1$$

$R_Y = \text{entire } z\text{-plane}$

$$\therefore R_X \cap R_H \subset R_Y$$

Initial Value Theorem (*for unilateral transforms*)

If $x[n]=x[n]u[n]$, i.e., $x[n]=0$ for $n<0$, then

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

To see this

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \left[\sum_{n=0}^{\infty} x[n]z^{-n} \right] = x[0]$$

All terms with $n>0$ become zero as $z^{-n} = 1/z^n \rightarrow 0$ as $z \rightarrow \infty$, except the first one which is always $x[0]$.

Final Value Theorem

$$\lim_{n \rightarrow \infty} f[n] = \lim_{Z \rightarrow 1} (1 - Z^{-1})F(Z)$$

$$\sum_{n=0}^{\infty} f(n)Z^{-n} - \sum_{n=0}^{\infty} f(n-1)Z^{-n}$$

Ex. Find the initial and final values of $f[n]$ where $F(z) = \frac{z}{z - .6}$.

$$F(z) = \frac{z}{z - .6}$$

Initial Value Thm (for $f[n] = 0 \ n < 0$)

$$x(0) = \lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \frac{1}{1 - .6/z}$$

$$= \underline{\underline{1}}$$

Final Value $\rightarrow 0$

Additional Properties of the z -Transform

Time Scaling Given

$$f[n] \stackrel{z}{\leftrightarrow} F(z),$$

show that

$$f\left[\frac{n}{k}\right] \stackrel{z}{\leftrightarrow} F(z^k)$$

for k a positive integer. This property can be seen by doing a change of variables (let $m = \frac{n}{k}$).

Scaling Time Domain

$$f\left[\frac{n}{a}\right] \leftrightarrow f(z^a)$$

Here a is a
positive integer

$$\sum_{n=-\infty}^{\infty} f\left[\frac{n}{a}\right] z^{-n}$$

$$\text{let } l = \frac{n}{a}$$

$$= \sum_{l=-\infty}^{\infty} f[l] z^{-al} = \sum_{l=-\infty}^{\infty} f[l] (z^a)^{-l}$$

$$= F(z^a)$$

Scaling Z Domain

$$a^n x(n) \leftrightarrow X\left(\frac{z}{a}\right)$$

Here a is a
real number

$$\begin{aligned} \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} &= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} \\ &= X\left(\frac{z}{a}\right) \end{aligned}$$

Properties of Z Transforms--Stability

As we saw earlier, a pole at a corresponds to a response of the form $a^k u[k]$ (for a right-sided signal)

So if $|a| > 1$, this value will grow large as k increases

And if $|a| < 1$ then it will go to zero as k increases.

Stable poles have magnitudes less than 1

- Poles and zeroes must come in complex conjugate pairs for the signal to be real

- Poles off the positive real axis correspond to an oscillating signal where the frequency of oscillation is the angle from the positive real axis. (Poles on the negative real axis have an angle of π , so the frequency of oscillation is π , as in $(-1)^n$.)

- When the poles are ...
 - on the unit circle \Rightarrow sinusoidal functions with constant amplitude
 - *not* on the unit circle \Rightarrow sinusoidal functions with a decaying (or growing) envelope (rate of decay/growth depends on the distance from the pole to the origin).

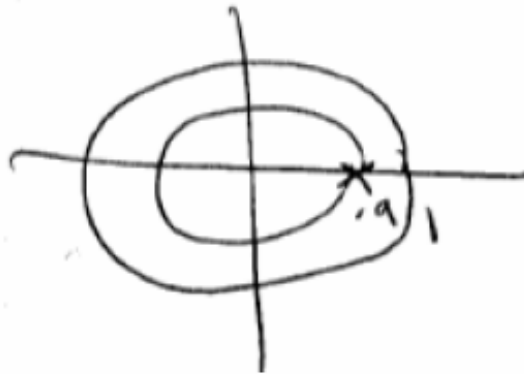
For bounded input, bounded output (BIBO) stability of a causal linear time invariant (LTI) discrete time systems, all poles of the transfer function $H(z)$ (that is, all roots of the characteristic equation) must lie within the unit circle in the z plane.

Because causal systems have Regions of Convergence that lie outside of the largest magnitude pole, an equivalent condition for BIBO stability is that the ROC must contain the unit circle

Ex. Find the z-transform of $x[n] = (.9)^n u[n]$. Would an LTI system with $x[n]$ as its system function be BIBO stable?

Yes

$$\frac{z}{z - .9}, \quad |z| > .9$$



Invertibility

$$h[n] * h_i[n] = \delta[n] \Rightarrow H(z)H_i(z) = 1$$

where $h_i[n] \xleftrightarrow{z} H_i(z)$ is the inverse of $h[n] \xleftrightarrow{z} H(z)$.

Ex. Find the inverse system $h_i[n]$ of $h[n] = a^n u[n]$.

Check your results by taking the convolution of $h[n]$ with $h_i[n]$.

$$H(z) = \frac{z}{z-a} \quad H_i(z) = \frac{z-a}{z} = 1 - az^{-1}$$

$$h_i[n] = \delta[n] - a\delta[n-1]$$

Ex. Find the inverse system of $h[n]$ where

$$H(z) = \frac{z-a}{z-b}$$

For BIBO stability of both systems (assuming they are both causal), where must all poles and zeros of $H(z)$ lie?

$$H_i(z) = \frac{z-b}{z-a} = \frac{z}{z-a} - \frac{b}{z-a}$$

$$h_i[n] = a^n u[n] - b a^{n-1} u[n-1]$$

Need all poles and zeros inside of the unit circle

Unilateral (one-sided) Z Transform

We can also define a **unilateral** z-transform (one-sided) by

*For causal impulse
response sequence*

$$H(z) \equiv \sum_{n=0}^{\infty} h[n]z^{-n}$$

By definition, the unilateral transform of any signal $x[n] = x[n]u[n]$ is equal to its bilateral transform.

When $x[n] \neq x[n]u[n]$ the two transforms are different.

Time Advance

$$\begin{aligned} Z(x[k+1]) &= \sum_{n=0}^{\infty} x[n+1]z^{-n} = z \sum_{m=1}^{\infty} x[m]z^{-m} \\ &= z \left[\sum_{m=0}^{\infty} x[m]z^{-m} - x[0] \right] = zX(z) - zx[0] \end{aligned}$$

$$m=n+1$$

Time Delay

$$\begin{aligned} Z(x[k-1]) &= \sum_{n=0}^{\infty} x[n-1]z^{-n} = z^{-1} \sum_{m=-1}^{\infty} x[m]z^{-m} \\ &= z^{-1} \left[\sum_{m=0}^{\infty} x[m]z^{-m} + zx[-1] \right] = z^{-1} X(z) + x[-1] \end{aligned}$$

$$m=n-1$$

Time Delay

$$\begin{aligned} Z(x[k-2]) &= \sum_{n=0}^{\infty} x[n-2]z^{-n} = z^{-2} \sum_{m=-2}^{\infty} x[m]z^{-m} \\ &= z^{-2} \left[\sum_{m=0}^{\infty} x[m]z^{-m} + zx[-1] + z^2x[-2] \right] = \\ &= z^{-2} X(z) + z^{-1}x[-1] + x[-2] \end{aligned}$$

$$m=n-2$$

Time Delay

In general,

$$Z(x[k - k_o]) = z^{-k_o} X(z) + \sum_{p=0}^{k_o-1} z^{-p} x[p - k_o]$$

Can use this formula to handle initial conditions