ME2025 Digital Control

Solving Difference Equations and Inverse Z Transforms

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Using Z Transforms To Solve Difference Equations

For LTI systems, described by linear constant coefficient difference equations

$$y[k]+a_1 y[k-1]+\ldots+a_n y[k-n] = b_0 x[k]+b_1 x[k-1]+\ldots+b_m x[k-m]$$

current and past outputs

current and past inputs

Closed Form Solutions to Difference Equations

Similar to the case for differential equations, difference equations have closed form solutions of the form

total response = zero-input component + zero-state component
= natural response + forced response
=complementary response + particular response

The Classical method for the solution is to express the output $y[n]$ as the sum of *complementary* or *natural* ($y_c[n]$) and *particular* or *forced* ($y_p[n]$) solutions:

$$y[n] = y_c[n] + y_p[n]$$

Using Z Transforms To Solve Difference Equations

For LTI systems, described by linear constant coefficient difference equations

$$y[k] + a_1 y[k-1] + \ldots + a_n y[k-n]$$
$$= b_0 x[k] + b_1 x[k-1] + \ldots + b_m x[k-m]$$

Take Z transform of both sides (or one sided Z transform assuming zero initial conditions):

Delays become powers of z^{-1}

$$Y(z)[1 + a_1 z^{-1} + \ldots + a_n z^{-n}] = X(z)[b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}]$$

Using Z Transforms To Solve Difference Equations

$$Y(z)[1+a_1z^{-1}+...+a_nz^{-n}] = X(z)[b_0+b_1z^{-1}+...+b_mz^{-m}]$$

Solve for the **Transfer Function H(z)** by dividing:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{[b_0+b_1z^{-1}+...+b_mz^{-m}]}{[1+a_1z^{-1}+...+a_nz^{-n}]}$$

$$= \frac{[b_0z^{n}+b_1z^{n-1}+...+b_mz^{n-m}]}{[z^{n}+a_1z^{n-1}+...+a_n]}$$

for $n \geq m$

Poles and Zeros

- Poles of H(z): roots of denominator polynomial
- Zeros of H(z): roots of numerator polynomial

  note: find these after canceling any common factors—and do this for polynomials in z (not z^{-1})

Using Z Transforms To Solve Difference Equations

- Find the output of an LTI system in the Z domain, *Y(z),* by multiplying the z-transform of the input, *X(z)* with *H(z)* = the Z transform of the impulse response
- Then you can use the Inverse Z Transform to get the output signal *y[k]* from its Z transform, *Y(z)*

Ex. Given a difference equation,

$$y[n] - .3y[n-1] = x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n] = (.6)^n u[n]$.

First step—take z Transforms of both sides of the equation.
Since x[n] is given, we can use the z-transform tables to substitute for X(z).

$$Y[z] - 0.3z^{-1}Y[z] = \frac{z}{z-.6}$$

X(z)

$$Y(z)[1 - 0.3z^{-1}] = \frac{z}{z-.6}$$

Factor out Y(z) –since we will want to inverse Z-transform it to get y[n]

Ex. Given a difference equation,

$$y[n]-.3y[n-1]=x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n]=(.6)^n u[n]$.

$$Y[z]-0.3z^{-1}Y[z]=\frac{z}{z-.6}$$

$$Y(z)[1-0.3z^{-1}]=\frac{z}{z-.6}$$

$$Y(z)=\left(\frac{z}{z-.06}\right)\left(\frac{z}{z-0.3}\right)$$

Now put everything in terms of z, rather than having z^{-1} terms—and solve for Y(z)

What if you wanted to find the response in the time domain?

$\Rightarrow$ We can use **Partial Fraction Expansion** to invert the z-transform.

Similar to what you saw for Laplace Transforms,

$$Y(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{N(z)}{D(z)}=\sum_{k=1}^{N}\frac{r_k z}{z-p_k}$$

where

$$p_k=\text{pole} \quad r_k=\text{residue}$$

For Distinct (non-repeated) roots

$$r_k=[\frac{Y(z)}{z}(z-p_k)]|_{z=p_k}$$

Then use tables to invert the z-transform, e.g.

$$a^n u[n]\leftrightarrow\frac{z}{z-a}$$

Returning to our example

$$y[n]-.3y[n-1]=x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n]=(.6)^n u[n]$.

$$Y[z]-0.3z^{-1}Y[z]=\frac{z}{z-.6}$$

$$Y(z)[1-0.3z^{-1}]=\frac{z}{z-.6}$$

$$Y(z)=\left(\frac{z}{z-.6}\right)\left(\frac{z}{z-0.3}\right)$$

$$\frac{Y(z)}{z}=\left(\frac{2}{z-.6}\right)+\left(\frac{-1}{z-0.3}\right)$$

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$$\frac{Y(z)}{z}=\left(\frac{2}{z-.6}\right)+\left(\frac{-1}{z-0.3}\right)$$

$$Y(z)=\left(\frac{2z}{z-.6}\right)+\left(\frac{-z}{z-0.3}\right)$$

↑ ↑

Now move the "saved" z back into the expression, so that these terms match things in the tables

Ex. Given a difference equation,

$$y[n]-.3y[n-1]=x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n]=(.6)^n u[n]$.

$$Y[z]-0.3z^{-1}Y[z]=\frac{z}{z-.6}$$

$$Y(z)[1-0.3z^{-1}]=\frac{z}{z-.6}$$

$$Y(z)=\left(\frac{z}{z-.6}\right)\left(\frac{z}{z-0.3}\right)$$

$$\frac{Y(z)}{z}=\left(\frac{2}{z-.6}\right)+\left(\frac{-1}{z-0.3}\right)$$

Finally, use tables to get *y[n]*

$$Y(z)=\left(\frac{2z}{z-.6}\right)+\left(\frac{-z}{z-0.3}\right)$$

$$y[n]=2(.6)^n u[n]-(.3)^n u[n]$$

From input — Natural response

One-sided Z Transform

- Key property—useful when considering initial conditions of difference equations:

$$Z(x[k+1]) = \sum_{n=0}^{\infty} x[n+1]z^{-n}$$

$$= \left(\sum_{n=0}^{\infty} x[n]z^{-(n-1)}\right) - x[0]z$$

$$= z\sum_{n=0}^{\infty} x[n]z^{-n} - x[0]z$$

$$= zX(z) - zx[0]$$

Example: for output *y* and input *u*

$y[k+1]-0.8y[k]=x[k]$ with initial condition $y[0]=2$

and unit step input $x[k]=u[k]$

Take the one-sided Z transform of equation and input to get

$$[zY[z]-zy[0]]-0.8Y[z]=\frac{z}{z-1}$$

Solve for *Y(z)* since we eventually want to get closed form solution *y[k]* for the difference equation.

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{z}{(z-1)(z-0.8)}$$

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{z}{(z-1)(z-0.8)}$$

Need to partial fraction the second term:

$$\frac{1 \text{ (not } z)}{(z-0.8)(z-1)}=\frac{k_1}{z-0.8}+\frac{k_2}{z-1}$$

$$k_1=\left.\frac{1}{z-1}\right|_{z=0.8}=\frac{1}{-0.2}=-5$$

$$k_2=\left.\frac{1}{z-0.8}\right|_{z=1}=\frac{1}{0.2}=5$$

So

$$\frac{z}{(z-0.8)(z-1)}=\frac{k_1 z}{z-0.8}+\frac{k_2 z}{z-1}$$

$$=\frac{-5z}{z-0.8}+\frac{5z}{z-1}$$

Hence

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{-5z}{(z-0.8)}+\frac{5z}{z-1}$$

$$Y[z] = y[0]\frac{z}{z-0.8} + \frac{-5z}{(z-0.8)} + \frac{5z}{z-1}$$

Inverse Z transforms give

$$y[k] = \{[y[0]-5](0.8)^k + 5\}u[k]$$

So with y[0] = 2,
get solution:

$$\boxed{y[k] = \{-3(0.8)^k + 5\}u[k]}$$

Ex. Find the Inverse z-Transform of

$$X(z) = \frac{2z^2 - 5z}{(z-2)(z-3)}, |z| > 3$$

↑ Right-sided

We begin by dividing out one "**z**" to protect it for later use in with out inverse **z** transform table

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)}$$

Ex. Find the Inverse z-Transform of

$$X(z) = \frac{2z^2 - 5z}{(z-2)(z-3)}, |z| > 3$$

Expand:

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)} = \frac{1}{z-2} + \frac{1}{z-3}$$

Next, bring back the "**z**" so that it matches the form in the table

$$X(z) = \frac{z}{z-2} + \frac{z}{z-3}$$

Ex. Find the Inverse z-Transform of

$$X(z) = \frac{2z^2 - 5z}{(z-2)(z-3)}, |z| > 3$$

Expand:

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)} = \frac{1}{z-2} + \frac{1}{z-3}$$

$$X(z) = \frac{z}{z-2} + \frac{z}{z-3}$$

$$\therefore x[n] = (2)^n u[n] + (3)^n u[n]$$

Right-sided

Ex. Find the Inverse z-Transform of

$$X(z)=\frac{2z^2-5z}{(z-2)(z-3)}, |z|>3$$

$$X(z)=\frac{z}{z-2}+\frac{z}{z-3}$$

$$\therefore x[n]=(2)^n u[n]+(3)^n u[n]$$

If the ROC had been $2<|z|<3$?

$$2^n u[n]-3^n u[-n-1]$$

If the ROC had been $|z|<2$?

$$-2^n u[-n-1]-3^n u[-n-1]$$

Ex. Given $h[n]=a^n u[n]$ ($|a|<1$) and $x[n]=u[n]$, find $y[n]=x[n]*h[n]$.

$$H(z)=\frac{z}{z-a} \qquad X(z)=\frac{z}{z-1}$$

$$\frac{Y(z)}{z}=\frac{z}{(z-a)(z-1)}=\frac{\frac{a}{a-1}}{z-a}+\frac{\frac{1}{1-a}}{z-1}$$

$$y[n]=\frac{a}{a-1}a^n u[n]+\frac{1}{1-a}u[n]$$

$$=\frac{1}{1-a}\left(1-a^{n+1}\right)u[n]$$

Complex Poles

$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_N z}{z - p_N}$$

$$x[n] = c_0\delta[n] + c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n, \quad n = 0, 1, 2, \ldots$$

Suppose $p_2 = \bar{p}_1$ then $c_2 = \bar{c}_1$.

So these 2 terms are $c_1 p_1^n + \bar{c}_1 \bar{p}_1^n$

We can write these terms as

$$2|c_1|\sigma^n \cos(\Omega n + \angle c_1)$$

$\sigma = |p_1|$ = magnitude of the pole p_1

$\Omega = \angle p_1$ = angle of p_1

$$p_1 = \sigma e^{j\Omega},\ c_1 = |c_1| e^{j\angle c_1}$$

$$p_1^n c_1 = \sigma^n |c_1| e^{j(\Omega n + \angle c_1)},\ \bar{p}_1^n \bar{c}_1 = \sigma^n |c_1| e^{j-(\Omega n + \angle c_1)}$$

$$p_1^n c_1 + \bar{p}_1^n \bar{c} = 2\sigma^n |c_1| \cos(\Omega n + \angle c_1)$$

So we have

$$x[n] = c_0\delta[n] + 2|c_1|\sigma^n \cos(\Omega n + \angle c_1) + c_3 p_3^n + \cdots + c_n p_N^n, n = 0, 1, 2, \ldots$$

The expression shows that, if $X(z)$ has a pair of complex poles p_1, p_2 with magnitude σ and angle $\pm\Omega$, the signal $x[n]$ contains a term of the form

$$2|c|\sigma^n \cos(\Omega n + \angle c)$$

$$X(z) = \frac{z^3 + 1}{z^3 - z^2 - z - 2}$$

then

$$A(z) = z^3 - z^2 - z - 2$$

$$p_1 = -0.5 - j0.866$$
$$p_2 = -0.5 + j0.866$$
$$p_3 = 2$$

Then, expanding $X(z)/z$ gives

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z + 0.5 + j0.866} + \frac{\bar{c}_1}{z + 0.5 - j0.866} + \frac{c_3}{z - 2}$$

where

$$c_0 = X(0) = \frac{1}{-2} = -0.5$$

$$c_1 = \left[(z + 0.5 + j0.866)\frac{X(z)}{z}\right]_{z=-0.5-j0.866} = 0.429 + j0.0825$$

$$c_3 = \left[(z - 2)\frac{X(z)}{z}\right]_{z=2} = 0.643$$

$$x[n] = -0.5\delta[n] + c_1(-0.5 - j0.866)^n + \bar{c}_1(-0.5 + j0.866)^n + 0.643(2)^n, \quad n = 0, 1, 2, \ldots$$

We can write the second and third terms in $x[n]$ in real form by using the form (7.68). Here, the magnitude and angle of p_1 are given by

$$|p_1| = \sqrt{(0.5)^2 + (0.866)^2} = 1$$

$$\angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3}\text{rad}$$

and the magnitude and angle of c_1 are given by

$$|c_1| = \sqrt{(0.429)^2 + (0.0825)^2} = 0.437$$

$$\angle c_1 = \tan^{-1}\frac{0.0825}{0.429} = 10.89°$$

Then, rewriting $x[n]$ in the form (7.69) yields

$$x[n] = -0.5\delta[n] + 0.874\cos\left(\frac{4\pi}{3}n + 10.89°\right) + 0.643(2)^n, \quad n = 0, 1, 2, \ldots$$

Repeated Poles

Repeated Poles. Again, let $p_1, p_2, \ldots, p_N$ denote the poles of $X(z) = B(z)/A(z)$, and assume that all the p_i are nonzero. Suppose that the pole p_1 is repeated r times and that the other $N - r$ poles are distinct. Then, $X(z)/z$ has the partial fraction expansion

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \cdots + \frac{c_r}{(z - p_1)^r} + \frac{c_{r+1}}{z - p_{r+1}} + \cdots + \frac{c_N}{z - p_N}$$

Repeated Poles

In (7.70), $c_0 = X(0)$ and the residues $c_{r+1}, c_{r+2}, \dots, c_N$ are computed in the same way as in the distinct pole case. [See (7.64).] The constants $c_r, c_{r-1}, \dots, c_1$ are given by

$$c_r = \left[(z - p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_{r-1} = \left[\frac{d}{dz}(z - p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_{r-2} = \frac{1}{2!}\left[\frac{d^2}{dz^2}(z - p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$

$$\vdots$$

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{dz^i}(z - p_1)^r \frac{X(z)}{z}\right]_{z=p_1}$$