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A Robust Controller Design Method for a Flexible Manipulator with a Large Time Varying Payload and Parameter Uncertainties

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Abstract. A new robust controller design method is proposed to obtain a less conservative feedback controller and it is applied to a single-link flexible manipulator. The objective is to maximize the control performance guaranteeing the robust stability when regulating the tip position of the flexible manipulator in the presence of a large time-varying payload and parameter uncertainties such as stiffness and joint friction. A descriptor form representation, which allows separate treatment of payload uncertainty from other parametric uncertainties, is used to reduce the conservatism of the conventional robust control approaches. Uncertainty of the payload in the inertia matrix is represented by polytopic approach and the uncertain parameters in the damping and stiffness matrices are treated with descaling techniques. Using aforementioned techniques, the robust LQ controller design problem for a flexible manipulator based on the guaranteed cost approach is formulated. Then, the formulated problem is solved by LMIs.

Key words: descaling technique, descriptor form, flexible manipulator, parameter uncertainty, payload variation, polytopic approach, robust control.

1. Introduction

Ever since the robot manipulators were introduced in the automation industry, manipulators have been refined to have better energy efficiency, faster operation and higher payload to arm weight ratio. These technical goals have been achieved up to a certain level by designing the low inertia and stiff structure. However, the concept of low inertia and stiff structure is relative to the motion speed and the control accuracy. As the manipulator motion speed is increased and the higher positioning accuracy is required, the manipulator cannot be considered as a low inertia or rigid structure. Therefore, control of a flexible manipulator has received much attention in the past two decades. Generally, it has been attempted to control the joint motion as well as a certain number of vibration modes with joint actuators [3, 5, 9].

The dynamic effect of the payload is much larger in the lightweight flexible manipulator than in the conventional rigid manipulator. Also, the inaccurate estimation of the actuating joint friction degrades the vibration mode control. To solve this problem, some control methods have been presented, usually based on adaptive control schemes [6, 7]. However, these approaches are restricted to the cases where the system has fixed parameter uncertainties. To cover the time-varying uncertain systems, robust control techniques such as sliding mode controls [22] and H_∞ controls [1, 24] are introduced to flexible manipulators. However, such robust control techniques may not yield high performance when a large uncertain payload exists, because the inversion of the inertia matrix in transformation process from a dynamic equation to a state-space form results in information loss on the structure and magnitude of all uncertainties.

Recently, some researchers have reported that a descriptor form is useful for representation of the uncertain system [10, 11]. The descriptor form can represent differential equations of a dynamic system more effectively than the state-space form. Especially, the descriptor form can preserve the independent physical parameters of the uncertainty structure of the inertia matrix.

This paper proposes a descriptor form based on a robust LQ controller design method for a flexible manipulator, which has model uncertainties and large payload variations. The differential equation of a single-link flexible manipulator has been represented in a descriptor form to separate payload uncertainty from other parametric uncertainties. We treated the uncertainty of the payload in the left-hand side inertia matrix with a polytopic approach and the uncertain parameters in the right-hand side damping and stiffness matrices with a descaling technique. The polytopic approach [4] is effective to preserve the uncertainty structure information when the uncertainty matrices in the model depend affinely on uncertain parameters. The descaling technique [8, 14, 15, 19–21, 23] is a conventional approach in general robust control strategy, such as scaled H_∞ control. By treating the system uncertainties using these two techniques, we can avoid the conservatism in estimating the bound of uncertainties. The optimal solution is obtained easily and systematically by using the linear matrix inequality (LMI) method [4].

Furthermore, this design method can be extended to the decentralized controller design for multi-link manipulators because the effect of external links to an inner

Table I. Physical properties of a single-link flexible manipulator

Link	EI: stiffness (Nm^2)	11.85	H: thickness (m)	47.63E–4
	ρA: unit length mass (kg/m)	0.2457	L: length	1.1938
Tip mass	M_e: mass (kg)	0.5867	J_e: rot. inertia (kgm^2)	0.2787
Hub	I_h: rot. inertia (kgm^2)		0.016	

link can be considered as a payload variation, damping and stiffness uncertainty, and external disturbances.

This paper is organized as follows. A dynamic model of a single-link flexible manipulator is presented in Section 2. The conventional design methods are introduced in Section 3. Section 4 addresses the proposed approach. The problem is formulated in Section 5. In Section 6, to evaluate the controller performance, the proposed control scheme is applied to a single-link flexible manipulator, which has time-varying payload, joint friction and stiffness uncertainties. Section 7 presents conclusions and discussions.

2. Dynamic Model

2.1. Single-link Flexible Manipulator Model

The flexible manipulator model employed in this manuscript is the one used by Kwon and Book [2, 17, 18] for their initial experiments on a flexible-link robot. The single-link flexible manipulator having a planar motion is described as shown in Figure 1. The rotating inertia of the servomotor, the tachometer, and the clamping hub are modeled as a single hub inertia I_h. The payload is modeled as an end mass M_e and a rotational inertia J_e. The joint friction is included in the damping matrix. The system parameters in Figure 1 are referred to in Table I.

The closed form dynamic equation is derived to show the system parameter structure using the assumed mode method. Here, we use the resulting dynamic equation with generalized coordinates as follows:

$$[M]\ddot{q} + [D]\dot{q} + [K]q = [U]\tau, \quad q = \begin{Bmatrix} q_i \\ \vdots \\ q_n \end{Bmatrix} \quad \text{for } i, j = 0, 1, \ldots, n, \tag{1}$$

$$[M] = \begin{bmatrix} M_{ij} & \cdots \\ \vdots & \ddots \end{bmatrix},$$

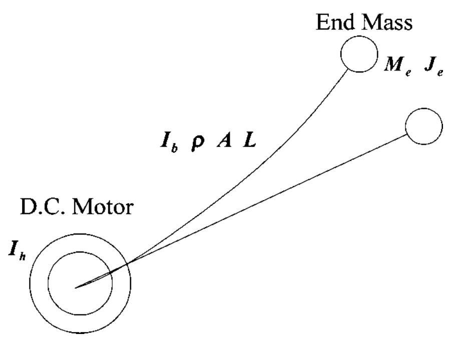


Figure 1. A single-link flexible manipulator.

$$M_{ij} = \rho A \int_0^l \Phi_i(x)\Phi_j(x)\,\mathrm{d}x + I_h\phi_i'(0)\Phi_j'(0) + M_e\Phi_i(l)\Phi_j(l) + J_e\Phi_i'(l)\Phi_j'(l),$$

$$[D] = c_0\begin{bmatrix} \Phi_i'(0)\Phi_j'(0) & \cdots \\ \vdots & \ddots \end{bmatrix}, \qquad [U] = \begin{bmatrix} \Phi_i'(0) \\ \vdots \end{bmatrix},$$

$$[K] = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & K_{ij} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad K_{ij} = EI\int_0^l \Phi_i''(x)\Phi_j''(x)\,\mathrm{d}x,$$

where $\Phi_{i,j}(\cdot)$ is a mode function, EI – stiffness of link, ρA – unit length mass of link, L – length of link, M_e – tip mass, J_e – tip rotational inertia, and I_h – rotational inertia of Hub.

For a state-space form, we obtain the following dynamic equation:

$$\dot{X} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} X + \begin{bmatrix} 0 \\ M^{-1}U \end{bmatrix} \tau = AX + B\tau, \tag{2}$$

where $X = \{q_r\ q_f\ \dot{q}_r\ \dot{q}_f\}^{\mathrm{T}} = \{q_0\ q_1\ \ldots\ \dot{q}_0\ \dot{q}_1\ \ldots\}^{\mathrm{T}}$, $q_r = q_0$ (rigid body coordinate),

$$q_f = \begin{Bmatrix} q_1 \\ \vdots \\ q_n \end{Bmatrix} \quad \text{(flexible mode coordinate).}$$

2.2. STRUCTURED UNCERTAINTY

If we assume that the tip-mass, tip rotational inertia, damping and bending stiffness have perturbations, the nominal system Equation (1) becomes an uncertain dynamic system as follows:

$$\big[M_0 + \Delta M(t)\big]\ddot{q} + \big[D_0 + \Delta D(t)\big]\dot{q} + \big[K_0 + \Delta K(t)\big]q = \big[U_0\big]\tau, \tag{3}$$

where M_0, D_0, K_0, and U_0 mean the nominal value,

$$[\Delta M] = \begin{bmatrix} \Delta M_{ij} & \cdots \\ \vdots & \ddots \end{bmatrix},$$

$$\Delta M_{ij} = \Delta M_e\Phi_i(l)\Phi_j(l) + \Delta J_e\Phi_i'(l)\Phi_j'(l),$$

$$[\Delta D] = \begin{bmatrix} \Delta D_{ij} & \cdots \\ \vdots & \ddots \end{bmatrix}, \qquad \Delta D_{ij} = \Delta c_0\Phi_i'(0)\Phi_j'(0),$$

$$[\Delta K] = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & \Delta K_{ij} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \Delta K_{ij} = \Delta EI\int_0^l \Phi_i''(x)\Phi_j''(x)\,\mathrm{d}x.$$

Here ΔM_e, ΔJ_e, Δc_0, and ΔEI mean perturbation of each parameter. Although these are scalar perturbations, the system has structured uncertainties as a multiplication of the mode function.

3. Conventional Design Methods

Until now, there have been numerous researches about a robust controller design method using a given structure information of uncertainties. Basically, there have been two main streams: one is the descaling technique [8, 14, 15, 19–21, 23] and the other is the polytopic approach [4]. In this section, these two approaches are explained briefly.

For a clear example, a simple class of uncertain systems is introduced

$$\dot{x} = A_\Delta x = \big(A_0 + \Delta A(t)\big)x, \tag{4}$$

$$z = Cx. \tag{5}$$

If the uncertainty matrix $\Delta A(t)$ depends affinely on the time varying parameters $\delta_i(t),\ i = 1, \ldots, r$, then the uncertainty in the system (4) can be expressed (without any loss of generality) as follows:

$$\Delta A(t) = \sum_{i=1}^{r} \delta_i(t) A_i = M_A \Delta^A(t) N_A. \tag{6}$$

The matrix A_i has a given uncertainty structure, and matrices M_A and N_A are used to describe the structure of uncertainty.

3.1. DESCALING TECHNIQUE

Descaling technique can be described by

$$\Delta A(t) = M_A \Delta^A(t) N_A = M_A \Gamma \Delta^A(t) \Gamma^{-1} N_A. \tag{7}$$

The uncertainty $\Delta A(t)$ can be decomposed into $M_A \Delta^A(t) N_A$ by using an input/output decomposition method, and design conservatism can be reduced using descaling matrix Γ [13, 14, 16, 20].

The descaling technique is the most popular approach in the robust control theory. And, it usually has been used in scaled H_∞-control theory. Using this approach, a tightly bounded stability condition can be obtained as a single constraint (such as a nonstandard algebraic Riccati equation [13, 14, 16, 20]). However, this approach becomes very conservative, when there exist uncertainties in an inertia term, because the inversion of an inertia matrix is inevitable when applying this technique.

For this reason, recently a polytopic approach has focused on the robust controller design.

3.2. POLYTOPIC APPROACH

Recall the simple uncertain system described by (4)–(6).

If the uncertainty matrix $\Delta A(t)$ is polytopic, i.e., it depends affinely on the time varying parameters $\delta_i(t),\ i = 1, \dots, r$, then the uncertain system can be expanded as a combination of the vertices in the variation range of uncertainty as follows:

$$A_0 + \Delta A(t) \in \{\alpha_1 A_1 + \cdots + \alpha_r A_r\colon\ \alpha_i \geqslant 0,\ \alpha_1 + \cdots + \alpha_r = 1\}.$$

If each system, whose system matrix is A_i, the vertex of the parameter variation range is stable, then the whole system that is represented by A_Δ in this set is stable. It means that it is sufficient to check the stability only on each vertex in order to show the stability of the time-varying system. This approach is effective in modeling the uncertainties when the uncertainties enter in the model affinely. This approach gives the same solution as the scaled H_∞-problem when the system has no inertia uncertainty. However, since it needs as many constraints as vertices, the more uncertain parameters give the longer computation time.

In the remainder of this section, a polytopic approach is explained in quadratic stability sense to facilitate the later expansion of the main theorem and the problem formulation. For simplicity, we shall also assume that the uncertainty matrix $\Delta^A(t) \in \Lambda^A$ is time-varying and the set Λ^A can be described as

$$\Lambda^A = \left\{\mathit{block\_diag}\left[\delta_1(t) I_{q_1}\ \dots\ \delta_r(t) I_{q_r}\right]\colon\ \underline{\delta}_i \leqslant \delta_i(t) \leqslant \overline{\delta}_i\right\},$$

where *block_diag* means a block diagonal matrix, $\underline{\delta}_i$ and $\overline{\delta}_i$ mean the minimum and maximum bounds of uncertainty variation range, respectively. For future reference, we shall denote the vertex set of Λ^A_{vex} with the extreme values as

$$\Lambda^A_{\mathrm{vex}} = \left\{\mathit{block\_diag}\left[\delta_1(t) I_{q_1}\ \dots\ \delta_r(t) I_{q_r}\right]\colon\ \delta_i(t) = \underline{\delta}_i \text{ or } \delta_i(t) = \overline{\delta}_i\right\}.$$

We can notice that there are 2^r vertices in Λ^A_{vex}.

The following lemma shows the usefulness of a polytopic approach when proving a quadratic stability of a time-varying system. This lemma is equivalent to Theorem 6 of [25], that is to a polytopic approach in quadratic stability sense.

LEMMA 1. *If an uncertain system is described by Equations* (3) *and* (4), *and the uncertainty matrix* $\Delta A(t)$ *is polytopic, the following two statements are equivalent:*

(i) *There exists a symmetric matrix* $P > 0$, *such that*

$$A_\Delta^{\mathrm{T}} P + P A_\Delta + C^{\mathrm{T}} C < 0 \quad \textit{for all } \Delta^A(t) \in \Lambda^A.$$

(ii) *There exists a symmetric matrix* $P > 0$, *such that*

$$A_\Delta^{\mathrm{T}} P + P A_\Delta + C^{\mathrm{T}} C < 0 \quad \textit{for all } \Delta^A(t) \in \Lambda^A_{\mathrm{vex}}.$$

Proof. The proof for (i) $\Rightarrow$ (ii) is trivial since $\Lambda^A_{\mathrm{vex}} \subset \Lambda^A$. To show (ii) $\Rightarrow$ (i), define $Q_\Delta \equiv A_\Delta^{\mathrm{T}} P + P A_\Delta + C^{\mathrm{T}} C$. Since Q_Δ depend affinely on $\Delta^A(t)$, it is easy to see by convexity that

$$\max_{\Delta \in \Lambda} \lambda_{\max}(Q_\Delta) = \max_{\Delta \in \Lambda_{\mathrm{vex}}} \lambda_{\max}(Q_\Delta).$$

This implies that $Q_\Delta < 0,\ \forall \Delta^A \in \Lambda^A$ if and only if $Q_\Delta < 0,\ \forall \Delta^A \in \Lambda^A_{\text{vex}}$. □

Using the polytopic approach, we can replace a stability constraint that should be satisfied in all parameter variation areas by another stability constraint that should be only satisfied in the vertices of the parameter variation area.

4. Proposed Approach

Two conventional robust controller design techniques, mentioned in Section 3, have both merit and demerit. This section explains the key idea of this paper to take only merits from these two techniques.

4.1. DESCRIPTOR-FORM REPRESENTATION

A descriptor-form representation makes it possible to use the two approaches simultaneously. The descriptor-form is as follows:

$$E\dot{x} = Fx + Hu, \tag{8}$$

where the matrix E contains the information of an inertia.

This form represents a system more intuitively (in contrast with a state-space form), and preserves independent physical parameter information, especially, for uncertain structure information on the inertia matrix.

In this approach, since a given uncertain dynamic equation is expressed in a descriptor-form, inertia uncertainties and other system uncertainties can be treated separately.

4.2. COMBINED APPROACH

If a general uncertain dynamic system is given, a descriptor-form of the general uncertain dynamic system is as follows:

$$\underbrace{(E_0 + \Delta E)\dot{x}}_{\text{Polytopic}} = \underbrace{(F_0 + \Delta F)x + (H_0 + \Delta H)u}_{\text{Descaling technique}}, \tag{9}$$

where E_0, F_0, and H_0 are nominal matrices and each Δ term means a structured uncertainty. If ΔE is considered to be polytopic, it is always possible to find a polytopic model as follows:

$$E_0 + \Delta E \in \{\alpha_1 E_1 + \cdots + \alpha_r E_r\colon\ \alpha_i \geqslant 0,\ \alpha_1 + \cdots + \alpha_r = 1\}.$$

We treat the left side matrix of Equation (9) (that describes inertia terms) in a polytopic approach, and manipulate these right side matrices (that are related to other dynamic terms) in a descaling technique. By using these two approaches simultaneously and avoiding the inversion of the inertia matrix, the bound of perturbation can be tightly obtained. The more detailed procedure will be expanded in the next section.

5. Problem Formulation

This section addresses a new robust LQ controller design scheme for a general uncertain system, which has a time-varying payload and parameter uncertainties in inertia, damping, stiffness and input matrices. Generally, the uncertain dynamic equation of a flexible manipulator can be described as follows:

$$(M_0 + \Delta M)\ddot{q} + (D_0 + \Delta D)\dot{q} + (K_0 + \Delta K)q = (U_0 + \Delta U)u, \tag{10}$$

where

$$\Delta M = \sum_{i=1}^{q_M} \delta_i(t) M_i, \qquad \Delta D = \sum_{j=1}^{q_D} \delta_j(t) D_j,$$
$$\Delta K = \sum_{k=1}^{q_K} \delta_k(t) K_k, \quad \text{and} \quad \Delta U = \sum_{l=1}^{q_U} \delta_l(t) U_l.$$

The real numbers $\delta_i, \delta_j, \delta_k$, and δ_l are uncertain and time-varying, and, without loss of generality, satisfy $|\delta_i| \leqslant 1$, $|\delta_j| \leqslant 1$, $|\delta_k| \leqslant 1$, and $|\delta_l| \leqslant 1$. The matrices M_i, D_j, K_k and U_l represent uncertainty structures.

The following Equation (11) is a descriptor-form of the uncertain dynamic equation (10) that maintains the uncertainty structure of the inertia matrix:

$$E_\Delta \dot{x} = (F_0 + \Delta F)x + (H_0 + \Delta H)u, \tag{11}$$
$$z = Cx + Du, \tag{12}$$

where $x = [q\ \dot{q}]^{\mathrm{T}} \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, z is the controlled output vector and C, D are weights of state and input, respectively.

$$E_\Delta = E_0 + \Delta E, \quad E_0 = \begin{bmatrix} I & 0 \\ 0 & M_0 \end{bmatrix}, \quad \Delta E = \begin{bmatrix} 0 & 0 \\ 0 & \Delta M \end{bmatrix},$$
$$F_0 = \begin{bmatrix} 0 & I \\ -K_0 & -D_0 \end{bmatrix}, \qquad H_0 = \begin{bmatrix} 0 \\ U_0 \end{bmatrix},$$
$$\Delta F = \begin{bmatrix} 0 & 0 \\ -\Delta K & -\Delta D \end{bmatrix}, \qquad \Delta H = \begin{bmatrix} 0 \\ \Delta U \end{bmatrix}.$$

Normally, the matrix E_Δ in the descriptor form (11) is assumed to be nonsingular and uncertainty matrix ΔM is polytopic. We can also define the compact set Λ^E, vertex set Λ^E_{vex} and Δ^E in (6), for uncertainty matrix ΔE.

The state-space form of (11) is as follows:

$$\dot{x} = \left(E_\Delta^{-1} F_0 + E_\Delta^{-1} \Delta F\right)x + \left(E_\Delta^{-1} H_0 + E_\Delta^{-1} \Delta H\right)u. \tag{13}$$

The LQ quadratic performance index is defined as $J_{\mathrm{LQ}} \equiv E[\int_0^\infty z^{\mathrm{T}} z \,\mathrm{d}t]$, where z is a controlled output, which was defined in (12).

The design objective of this problem is to find the full-state feedback controller $u = -Gx$, which stabilizes the uncertain system (13) and minimizes the time domain LQ performance index J_{LQ}. However, it is impossible to minimize directly the performance index J_{LQ} stabilizing the time-varying system (13). Thus, a guaranteed cost control approach [14], which minimizes the upper bound of the performance indices, is applied.

If there exists a Lyapunov function $V(x) = x^{\mathrm{T}} P x$ that satisfies the inequality

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} + z^{\mathrm{T}} z < 0 \tag{14}$$

for $\forall \Delta^E \in \Lambda^E$, then the system (13) is quadratically stable and the quadratic performance index is bounded by *trace*$[PX(0)]$, where $X(0) = E[x(0)\, x(0)^{\mathrm{T}}]$. This performance bounding result can be obtained easily by integrating Equation (14). If (14) is expanded using (13) and $u = -Gx$, the following quadratic stability constraint is obtained:

$$\begin{aligned}
&\big(E_\Delta^{-1}(F_0 - H_0 G)\big)^{\mathrm{T}} P + P E_\Delta^{-1}(F_0 - H_0 G) + (C - DG)^{\mathrm{T}}(C - DG) + \\
&\quad + \big(E_\Delta^{-1}\Delta F\big)^{\mathrm{T}} P + P E_\Delta^{-1}\Delta F - \big(E_\Delta^{-1}\Delta H G\big)^{\mathrm{T}} P - P E_\Delta^{-1}\Delta H G < 0 \\
&\quad \text{for } \forall \Delta^E \in \Lambda^E.
\end{aligned} \tag{15}$$

Now, the design problem becomes an optimization problem that minimizes the upper bound of the performance indices, *trace*$[PX(0)]$, subject to the inequality constraint (15). However, it is difficult to find the controller gain G, satisfying condition (15), because this constraint is nonlinear and time-varying. Moreover, since the inversion of the uncertain inertia matrix loses the structure and magnitude information of all uncertainties, this constraint becomes very conservative.

From now on, we transform this complex inequality constraint into a linear matrix form to solve this optimization problem by LMI. If the optimization problems can be transformed into LMI problems, then the convex global optimization is guaranteed by efficient search algorithms such as interior-point method and ellipsoid algorithm [4]. In this transformation process, a tightly bounded stability constraint can be obtained by avoiding the inversion of the uncertain inertia matrix. The uncertainties of the system and input matrices can be changed via I/O factorization technique [16] as follows:

$$\begin{aligned}
E_\Delta^{-1}\Delta F &= \big(E_\Delta^{-1} M_F \Gamma_F\big)\Delta^F\big(\Gamma_F^{-1} N_F\big), \\
E_\Delta^{-1}\Delta H &= \big(E_\Delta^{-1} M_H \Gamma_H\big)\Delta^H\big(\Gamma_H^{-1} N_H\big).
\end{aligned} \tag{16}$$

Sequentially, by multiplying symmetric matrices P^{-1} and E_Δ to both sides of (15), the quadratic stability constraint (15) is transformed as follows:

$$\begin{aligned}&E_\Delta P^{-1}F_0^{\mathrm{T}} + F_0 P^{-1}E_\Delta^{\mathrm{T}} - E_\Delta P^{-1}G^{\mathrm{T}}H_0^{\mathrm{T}} - H_0 G P^{-1}E_\Delta^{\mathrm{T}} +\\ &\quad + E_\Delta P^{-1}(CZ-DG)^{\mathrm{T}}(CZ-DG)P^{-1}E_\Delta^{\mathrm{T}} + \left(M_F\Gamma_F\Delta^F\Gamma_F^{-1}N_F P^{-1}E_\Delta\right)^{\mathrm{T}} +\\ &\quad + M_F\Gamma_F\Delta^F\Gamma_F^{-1}N_F P^{-1}E_\Delta - \left(M_H\Gamma_H\Delta^H\Gamma_H^{-1}N_H G P^{-1}E_\Delta\right)^{\mathrm{T}} -\\ &\quad - M_H\Gamma_H\Delta^H\Gamma_H^{-1}N_H G P^{-1}E_\Delta < 0 \quad \text{for } \forall\Delta^E \in \Lambda^E.\end{aligned} \tag{17}$$

Substituting $P^{-1} = Z,\ GP^{-1} = Y$, we see that constraint (17) is transformed equivalently as follows:

$$\begin{aligned}&E_\Delta Z F_0^{\mathrm{T}} + F_0 Z E_\Delta^{\mathrm{T}} - E_\Delta Y^{\mathrm{T}}H_0^{\mathrm{T}} - H_0 Y E_\Delta^{\mathrm{T}} +\\ &\quad + E_\Delta(CZ-DY)^{\mathrm{T}}(CZ-DY)E_\Delta^{\mathrm{T}} +\\ &\quad + \left(M_F\Gamma_F\Delta^F\Gamma_F^{-1}N_F Z E_\Delta\right)^{\mathrm{T}} + M_F\Gamma_F\Delta^F\Gamma_F^{-1}N_F Z E_\Delta -\\ &\quad - \left(M_H\Gamma_H\Delta^H\Gamma_H^{-1}N_H Y E_\Delta\right)^{\mathrm{T}} - M_H\Gamma_H\Delta^H\Gamma_H^{-1}N_H Y E_\Delta < 0\\ &\qquad \text{for } \forall\Delta^E \in \Lambda^E.\end{aligned} \tag{18}$$

Applying an uncertainty bounding technique [16, 25] and using the scaling matrices, we get

$$\begin{aligned}&X_F \in S_{X_F}\left(\equiv \left\{\Gamma_F\Gamma_F^{\mathrm{T}} \mid \Gamma_F \in S_{\Gamma_F}\right\}\right) \quad \text{and}\\ &X_H \in S_{X_H}\left(\equiv \left\{\Gamma_H\Gamma_H^{\mathrm{T}} \mid \Gamma_H \in S_{\Gamma_H}\right\}\right);\end{aligned}$$

the quadratic stability criteria (18) is bounded as

$$\begin{aligned}&E_\Delta Z F_0^{\mathrm{T}} + F_0 Z E_\Delta^{\mathrm{T}} - E_\Delta Y^{\mathrm{T}}H_0^{\mathrm{T}} - H_0 Y E_\Delta^{\mathrm{T}} + M_F X_F M_F^{\mathrm{T}} + M_H X_H M_H^{\mathrm{T}} +\\ &\quad + E_\Delta(CZ-DY)^{\mathrm{T}}(CZ-DY)E_\Delta^{\mathrm{T}} + E_\Delta^{\mathrm{T}} Z N_F^{\mathrm{T}} X_F^{-1} N_F Z E_\Delta +\\ &\quad + E_\Delta^{\mathrm{T}} Y^{\mathrm{T}} N_H^{\mathrm{T}} X_H^{-1} N_H Y E_\Delta < 0 \quad \text{for } \forall\Delta^E \in \Lambda^E.\end{aligned} \tag{19}$$

Using the Schur complement [4], the matrix inequality constraint is obtained as follows:

$$\begin{bmatrix} \begin{pmatrix} E_\Delta Z F_0^{\mathrm{T}} + F_0 Z E_\Delta^{\mathrm{T}} - \\ -E_\Delta Y^{\mathrm{T}} H_0^{\mathrm{T}} - H_0 Y E_\Delta^{\mathrm{T}} + \\ +M_F X_F M_F^{\mathrm{T}} + \\ +M_H X_H M_H^{\mathrm{T}} \end{pmatrix} & E_\Delta(CZ-DY)^{\mathrm{T}} & E_\Delta Z N_F^{\mathrm{T}} & E_\Delta Y^{\mathrm{T}} N_H^{\mathrm{T}} \\ (CZ-DY)E_\Delta^{\mathrm{T}} & -I & & \\ N_F Z E_\Delta^{\mathrm{T}} & & -X_F & \\ N_H Y E_\Delta^{\mathrm{T}} & & & -X_H \end{bmatrix} < 0$$
$$\text{for } \forall\Delta^E \in \Lambda^E. \tag{20}$$

Because the above constraint should be satisfied $\forall\Delta^E \in \Lambda^E$, that is, for all variation ranges of E_Δ, it is difficult to find the design variables Z, Y, X_H, and X_F which satisfy the matrix inequality constraint (20). However, we can find that constraint (20) is affine for E_Δ. Since Δ^E appears to be affine in the above constraint, inequality constraint (20) is valid in the range of $\forall\Delta^E \in \Lambda^E_{\text{vex}}$ by Lemma 1:

$$\begin{bmatrix} \begin{pmatrix} E_\Delta Z F_0^{\mathrm{T}} + F_0 Z E_\Delta^{\mathrm{T}} - \\ -E_\Delta Y^{\mathrm{T}} H_0^{\mathrm{T}} - H_0 Y E_\Delta^{\mathrm{T}} + \\ +M_F X_F M_F^{\mathrm{T}} + \\ +M_H X_H M_H^{\mathrm{T}} \end{pmatrix} & E_\Delta (CZ - DY)^{\mathrm{T}} & E_\Delta Z N_F^{\mathrm{T}} & E_\Delta Y^{\mathrm{T}} N_H^{\mathrm{T}} \\ (CZ - DY) E_\Delta^{\mathrm{T}} & -I & & \\ N_F Z E_\Delta^{\mathrm{T}} & & -X_F & \\ N_H Y E_\Delta^{\mathrm{T}} & & & -X_H \end{bmatrix} < 0$$

$$\text{for } \forall\Delta^E \in \Lambda^E_{\text{vex}}. \tag{21}$$

Thus, we say that uncertain system (13) is quadratically stable if we can find symmetric matrices $Z > 0$, $X_F > 0$, $X_H > 0$ and properly dimensioned Y that satisfies matrix inequality condition (21) only for all vertices in Λ^E_{vex}, simultaneously. The full state feedback controller is taken as $G = YZ^{-1}$.

Note that the equality constraint $Z = P^{-1}$ is used in transformation (18). However, this equality constraint is nonconvex. Thus, the equality constraint $Z = P^{-1}$ should be replaced by a convex inequality constraint [7]

$$Z - P^{-1} > 0 \quad \text{or} \quad \begin{bmatrix} Z & I \\ I & P \end{bmatrix} > 0. \tag{22}$$

In this section, we have shown that the design problem of the robust LQ regulator can be reduced to an optimization problem searching for the matrices $Z > 0$, $X_F > 0$, $X_H > 0$ and properly dimensioned Y minimizing *trace*$[PX(0)]$ while satisfying the linear matrix inequality constraints (21) and (22). Since these matrix inequalities are convex in Z, X_F, X_H, and Y, the convex programming techniques can be used to find Z, X_F, X_H and Y.

6. Numerical Simulation

In the previous Section 5, the robust LQ controller design method was described for a general uncertain system. In this section, the robust LQ regulator is designed and the performance of the controller is verified for the single-link flexible manipulator that was used in [17]. We consider the case where an additional mass is attached to the end of the link. Also, the joint friction coefficient and the stiffness of the link are assumed to have parameter variations.

In Section 2, we described the nominal model and parameter uncertainties of the single-link flexible manipulator. These uncertainties in Equation (3) can be normalized as follows:

$$\Delta M_{ij} = \delta_1(t)\big(\max(\Delta M_e)\Phi_i(l)\Phi_j(l) + \max(\Delta J_e)\Phi_i'(l)\Phi_j'(l)\big), \tag{23}$$

$$\Delta D_{ij} = \delta_2(t)\max(\Delta c_0)\Phi_i'(0)\Phi_j'(0), \tag{24}$$

$$\Delta K_{ij} = \delta_3(t)\max(\Delta EI)\int_0^l \Phi_i''(x)\Phi_j''(x)\,\mathrm{d}x, \tag{25}$$

where $|\delta_1(t)| \leqslant 1,\ |\delta_2(t)| \leqslant 1,\ |\delta_3(t)| \leqslant 1$.

Notice that ΔM_e and ΔJ_e are dependent to each other. Thus, these are assumed to be expressed as one parameter uncertainty $\delta_1(t)$. We can also say that the uncertainty matrices $\Delta M,\ \Delta D$, and ΔK are polytopic, i.e., they depend affinely on the time-varying parameters $\delta_1(t), \delta_2(t)$, and $\delta_3(t)$, respectively. δ_1, δ_2, and δ_3 represent normalized variations of inertia, damping and stiffness uncertainty, respectively.

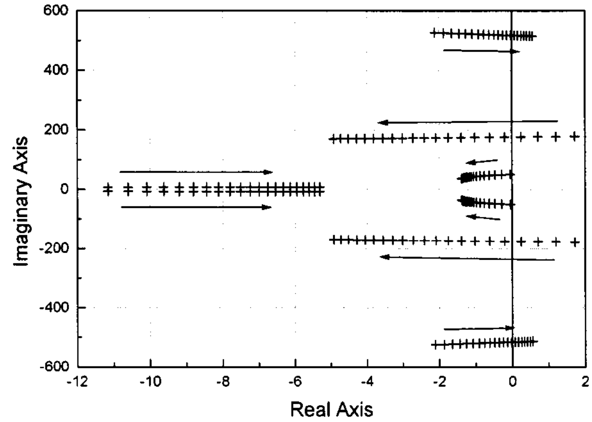
The tip mass and the tip rotational inertia are assumed to have a variation of ±40% from the nominal values during operation, and the damping and stiffness matrix of the link have fixed perturbation within ±50% and ±30% from the nominal values, respectively. For the system dynamic model, the flexible mode is modeled up to the third mode, that is, the 8th order system is considered.

In order to compare the performance, the proposed robust LQ regulator, conventional robust LQ regulator, and nominal LQ regulator have been designed with the same weighting function of the performance index (12), using the following matrices:

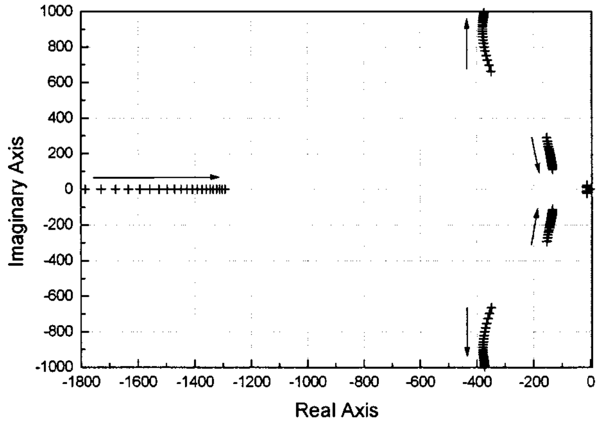
$$C = diag[5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1 \quad 0 \quad 0],$$
$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{0.1} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

To understand the effect of uncertainties of the system, the closed-loop poles have been traced while the tip mass and the tip rotational inertia varies ±40% from the nominal values with the actual damping and stiffness matrix of the link that have fixed perturbation of +50% and −30% from the nominal values, respectively. Figure 2(a)–(c) show the closed-loop system poles of the nominal LQ regulator, conventional robust LQ regulator and proposed robust LQ regulator, respectively. For the given weighting matrices C and D, a nominal LQ regulator is designed to locate its poles near the imaginary axis. As a result, its closed-loop poles cross to the RHP for some parameter perturbations. On the other hand, since the conventional robust LQ regulator and the proposed LQ regulator is designed taking into account the parameter perturbations, the closed-loop poles of these robust controllers remain stable for the same parameter perturbations. The closed-loop poles of the conventional robust LQ regulator are located further to the left than the ones of the proposed robust LQ regulator. It is clear that the conventional robust LQ regulator is designed more conservatively than the proposed controller. Moreover, the range of the pole loci of the proposed robust LQ regulator is smaller that that of the conventional robust LQ regulator for the end-mass perturbation. That means that the proposed LQ regulator is more insensitive to the parameter variations.

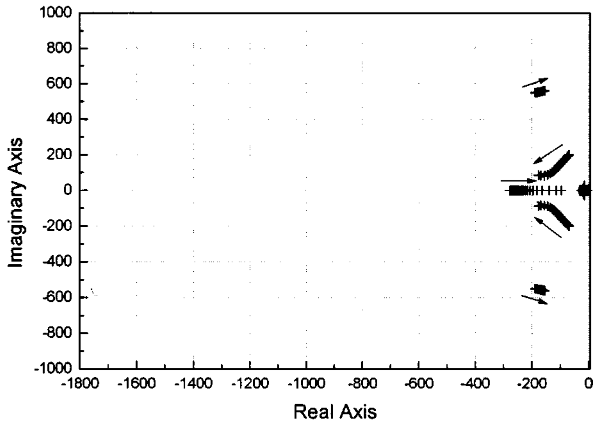
In order to show the performance improvement of the proposed robust LQ regulator, the regulation problem has been simulated with the proposed controller and the conventional controller. Figure 3 compares the Hub angle, the tip position and the control input of these two controllers for the following system parameter uncertainties. We considered the system with −50% stiffness and +30% damping perturbation, and the tip mass and the tip rotational inertia having a sinusoidal



(a) Nominal LQ regulator

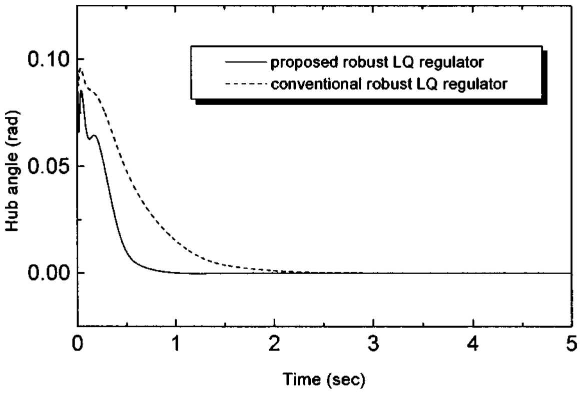


(b) Conventional Robust LQ regulator

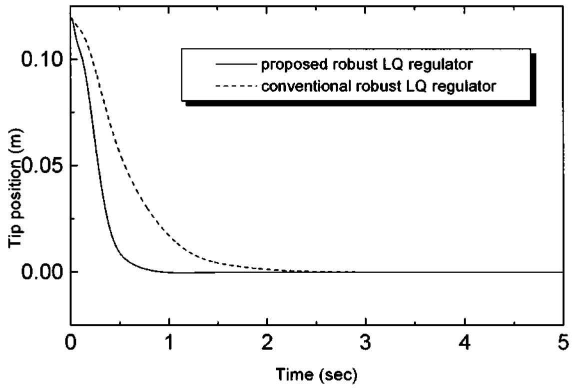


(c) Proposed Robust LQ regulatorController

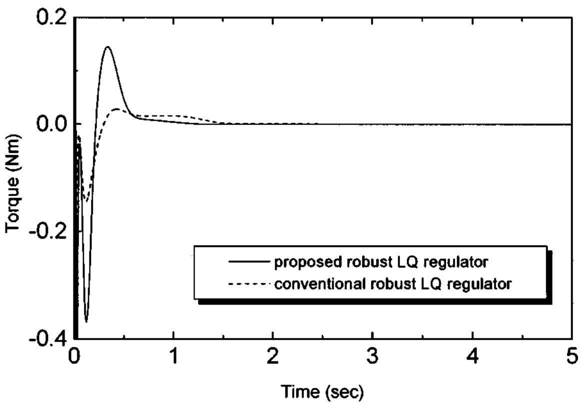
Figure 2. Location of poles for the end-mass variation (Closed-loop system).



(a) Hub angle



(b) Tip Position



(C) Control Input

Figure 3. Comparison of control performance.

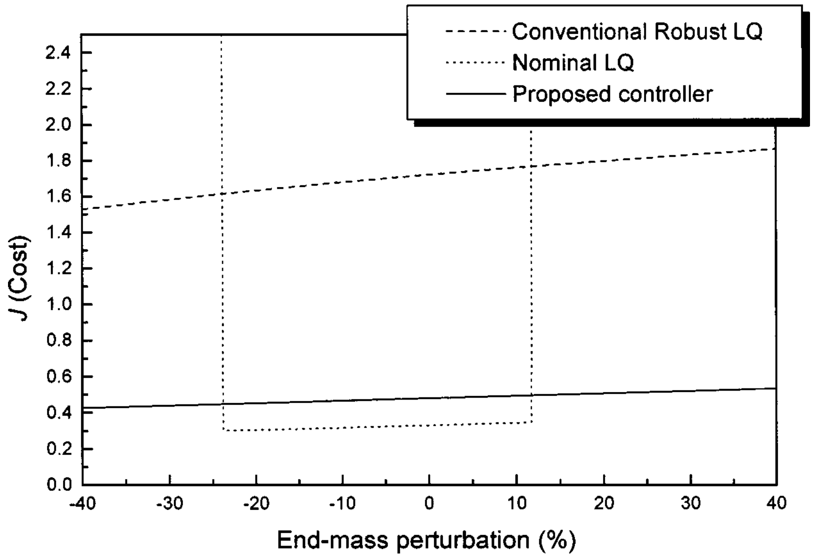


Figure 4. Comparison of performance for the end-mass perturbation.

change of ±40% ($\delta_1 = \pm 1$) from the nominal values. The proposed controller shows better performance than the conventional robust LQ controller. Moreover, the proposed robust LQ controller demands less control input than the conventional robust LQ controller, especially in the initial stage. The initial control effort of each controller is -8.8235 Nm (proposed controller) and -62.777 Nm (conventional controller). Figure 3(c) shows the zoom view of the control input. To compare the control efforts more precisely, the integral measure of the control input is brought as $\mathrm{Im} = \int_0^t |u|\,\mathrm{d}t$. The proposed controller and the conventional robust LQ controller's integral measure is 0.1553 and 0.6674, respectively. The result shows that the conventional robust LQ regulator is conservatively over-designed.

The performance indices J_{LQ} of each controller are shown for the end-mass perturbation in Figure 4. The proposed controller and the conventional robust LQ regulator, which are used in simulation, are designed to be stable for the same amount of parameter perturbation. Even though these two controllers remain stable, the performance of the proposed robust LQ regulator is improved in comparison with the conventional Robust LQ regulator. The nominal LQ regulator shows the best performance for the system without any end-mass perturbation. However, it cannot guarantee stability for the large amount of end-mass perturbation.

Notice that the end-mass perturbation limits where the conventional robust LQ regulator guarantees stability are ±40% from the nominal value in the presence of +50% damping and −30% stiffness perturbation. On the other hand, the proposed regulator extends this amount of the end-mass perturbation to ±80% from the nominal value. Thus, the proposed controller can stabilize the wider range of parameter perturbation due to the conservatism of the controller design scheme.

If we consider the multi-link case, the dynamic equation of any link can be expressed by adding the dynamic effects of the outer and the inner links with the additive of structured uncertainties and unstructured disturbance. These structured uncertainties, which depend on configuration changes of other links, exist on inertia, damping and stiffness terms, simultaneously. The disturbance represents interaction, centrifugal, coriolis and gravity forces. Since the proposed controller is effective for the structured uncertain system with inertia, damping, stiffness and input perturbation, it is expected that the proposed robust LQ controller design method can be applied to the decentralized controller design for multi-link flexible manipulators.

7. Conclusions

A new design method for the robust LQ regulator is proposed. It is based on the descriptor form for the control of a single-link flexible manipulator, which has a large uncertain payload variation and structured system parameter uncertainties. By using the descriptor form representation, the inversion of the inertia matrix is avoided. Thus, tightly bounded stability constraint is obtained by maintaining the inertia matrix uncertainty structure. We have designed a less conservative robust controller than the conventional controller that is designed by using only the descaling technique, by applying the descaling technique and the polytopic approach simultaneously. Also, the proposed design method is more practical than the design method which uses only polytopic approach, because a number of design constraints are reduced. The controller design problem is formulated as a convex programming problem and is easily solved using LMIs. As a result, the controller designed with the proposed method shows the improved robust performance and the reduced conservatism.

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